REALIZATION OF LINEAR DISCRETE TIME-VARYING SYSTEMS

By

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This work is concerned with the construction of linear, discrete, time-varying, finite-dimensional state-variable models (a realization) from input/output data of discrete dynamical systems. The input/output data are represented by a linear map f_W with a pxm matrix W(k, l) as its unit-pulse response. Local and global conditions for realizability are given in terms of the notion of the generalized Hankel matrix. An algorithm to construct a canonical realization, where it is possible for one to exist, is developed. In the last part of this work, the realization algorithm is applied to the problem of system reduction.

CHAPTER ONE
INTRODUCTION

A discrete time-varying dynamical system is usually described in two ways: by a state-space description, and by an input/output (i/o) description. The state description is generally used when one is interested in the internal behavior of a system and it consists of a first-order vector difference equation $x(k+1) = F(k)x(k) + G(k)u(k)$, and an output equation $y(k) = H(k)x(k)$ where $x(k)$ is the state, $u(k)$ is the input, and $y(k)$ is the output. On the other hand, the i/o description arises naturally when one considers the system's external behavior,and it is defined here by a linear input/output map whose "kernel" or "unit-pulse response function" is a pxm matrix $W(k,l)$ with $k > l$. It is also possible to describe a time-varying dynamical system from the input/output point of view by its "transfer function" (see KAMEN and KHARGONEKAR [1982] and POOLLA [1984] for details). We shall not consider in this work the transfer function approach. The state-space and input/output approaches, far from being mutually exclusive, are both used in control problems.

The translation from an input/output description to a state-space one is an important question. In particular, the problem of finding a state model for a given system described by its i/o map is called the

problem of realization. Exactly how one does this in the case of continuous time-varying system has been the subject of many papers in recent years (see SILVERMAN and MEADOWS [1966], KALMAN [1968b], KAMEN [1979], FLIESS [1982, 1983]). Surprisingly,this question has received little attention in the case of discrete time-varying systems.

It turns out that the solution of the problem of realization in the discrete-time framework is much more than a simple carry-over of the corresponding one for continuous-time systems. In this work, we shall see that there exist basic differences between the continuous-time and the discrete-time frameworks. Let us suppose we want to analyze a "black box" system Σ on the basis of input/output experiments. Then, strictly speaking, the impulse response matrix $W_\Sigma(t, \tau)$, when Σ is continuous, and the pulse-response matrix $W_\Sigma(k, l)$, when Σ is discrete, can only be determined (or measured) over the causal half plane ($t \geq \tau$ and $k \geq l$ respectively). In the continuous-time case, when the entries of W_Σ are analytic functions, the impulse response can be extended uniquely to the region $t < \tau$ by analytic continuation to obtain what is called the "weighting pattern" matrix, $\bar{W}_\Sigma$, of the system. For a discrete-time system, however, there is no unique extension of $W_\Sigma(k,l)$ to the region $k < l$ due to the discrete-time nature of the system. Since most of the existing realizability criteria for continuous i/o maps are based on the notion of the weighting pattern, $\bar{W}_\Sigma$, one should not expect to be able to carry over those results to the discrete-time case.

The problem of realization for linear discrete time-varying systems was first considered by WEISS [1972] and EVANS [1972]. The former

author gave a realization procedure for linear input/output maps known to have canonical realizations (that is, state-space models which are both reachable and observable); however, he did not give realizability criteria nor did he determine those input/output maps which have a canonical realization. On the other hand, Evans did not provide a realization scheme even though he derived necessary and sufficient conditions for realizability. Unfortunately, Evans's criteria are not easy to implement from the computational point of view.

As noted above, the problem of realization for discrete time-varying input/output maps is far from complete. In this dissertation, the discrete-time realization problem is approached from an entirely different angle from that employed by Weiss and Evans (though some results due to the latter author are used).

The major contributions of this project are

(i) the introduction of the notion of representative formal power series and the concept of the (generalized) Hankel matrix in the study of realizability of linear discrete i/o maps;

(ii) the discovery of not only "local" realizability criteria but also of "global" ones which can be considered as the discrete-time counterpart of those obtained by KALMAN [1968b] and by KAMEN [1979] respectively;

(iii) the introduction of the concept of response maps with finite constant rank, the importance of which maps is shown by the fact that they are the only kind for which there exists a canonical realization;

(iv) the development of an algorithm to find canonical realizations.

It should be noted that the results obtained in this project relate only to the noise-free case.

Chapter Two of the report gives a summary of realization theory for time-invariant systems and for continuous time-varying ones in order to provide the reader with the necessary background and to facilitate the comparison of those systems with the discrete time-varying ones. Here, the problem of realization in the discrete-time framework is stated.

In Chapter Three, we consider a local realizability criterion which is obtained through the theory of generalized representative formal power series. The scalar version of this kind of series was used for the first time by FLIESS [1982, 1983] in his work on realization of real analytic time-varying systems. This approach to the realization problem (in the discrete-time framework) is believed to be novel. Furthermore, it is important to say that the principal result of this section is expressed in terms of the Hankel matrix, and it can be considered as the discrete-time counterpart of those results obtained by KALMAN [1968], SILVERMAN and MEADOWS [1966], and SILVERMAN [1971] for continuous-time systems.

In Chapter Four, we find realizability criteria which require that the unit pulse response of a given i/o map satisfies some recursive relation. These conditions are the discrete-time version of the ones developed in KAMEN [1979] for real analytic time-varying systems.

In Chapter Five, we determine some global realizability criteria based on the concept of the Hankel matrix. In the framework developed here, the Hankel matrix is defined over the ring B of all bi-infinite sequences over a fixed field K. Realizability criteria are given in terms of the finiteness of modules over B spanned (or generated) by the elementary rows or columns of the Hankel matrix. These results are similar to realizability criteria for linear real-analytic time-varying systems developed by KAMEN [1979].

In Chapter Six we introduce the concept of linear input/output maps with finite-constant rank, and show that only for this kind of response maps it is possible to construct realizations which are both reachable and observable (i.e., canonical). Also, it is seen here that it is not true that the realization theory for discrete-time systems is a simple carry over of the corresponding one for continuous-time systems. In fact, we show that the i/o map of a constant rank system does not necessarily have finite constant-rank (for continuous-time systems it is true though, see KAMEN [1979]).

In Chapters Seven and Eight, the construction of canonical (minimal) realizations for response maps with finite-constant rank is considered. By using a method for computing a basis for a finite free module defined over some subrings of B, we develop a realization procedure for computing canonical realizations. This algorithm is the discrete-time counterpart of the one given in KAMEN [1979] for real analytic time-varying systems, and it is a generalization of WEISS's algorithm.

In Chapter Eight, we also solve the important problem of getting a canonical realization, if one exists, from a given realization $\Sigma = (D, E, H)$ of a linear i/o map f_Σ.

In Appendices One and Two, we review some algebraic results which will be used in Chapters Seven and Eight. We also propose a more general version of the problem of realization studied in this dissertation. Furthermore, we find necessary and sufficient conditions for realizability of input/output maps defined on some "proper" subsets of the causal half-plane.

CHAPTER TWO
SYSTEM DEFINITIONS AND REVIEW OF SOME REALIZABILITY RESULTS

In this section, some basic definitions and a summary of realization theory of linear systems will be given in order to help the reader to understand better the problem to be solved in this project.

2.1 Time-invariant systems

For any fixed field K, and for any fixed $n \in Z^+$ = set of all positive integers, let us denote by K^n the K-linear space consisting of all n-element column vectors whose entries are in K.

An m-input p-output n-dimensional time-invariant system Σ over K is a triple $\Sigma := (F,G,H)$ of nxn, nxm, pxm matrices over K together with the following dynamical equations

$$\begin{aligned} (\Theta x)(t) &= F\,x(t) + G\,u(t); \\ y(t) &= H\,x(t) \qquad\qquad , \; t \in T \end{aligned} \tag{2.1.1}$$

where T = (ordered) additive subset of $\mathbf{R}$ (the field of real numbers), $x(t) \in K^n$ is the state of the system at time t, $u(t) \in K^m$ is the input or control law at time t, $y(t) \in K^p$ is the output at time t, and Θ is an operator which can be interpretated as

(i) the left shift operator, i.e., $(\theta x)(t) = x(t+1)$; in this case, T is usually taken as $Z^+ \cup \{o\}$, and Σ is called a discrete time-invariant system with state equations

$$\begin{aligned} x(t+1) &= F\,x(t) + G\,u(t), \\ y(t) &= H\,x(t), \end{aligned} \tag{2.1.2}$$

(ii) the derivative operator; that is $(\theta x)(t) = \frac{d}{dt}x(t)$. Then, we assume that $K = \mathbf{R}$, and we say that Σ is a continuous time-invariant system with state equations

$$\begin{aligned} \frac{d}{dt}x(t) &= F\,x(t) + G\,u(t), \\ y(t) &= H\,x(t). \end{aligned} \tag{2.1.3}$$

For $x(o) = o$, the solution of (2.1.2) can be shown to be

$$y(t) = \sum_{\ell=o}^{t-1} H\,F^{t-\ell-1}\,G\,u(\ell), \quad \text{for all } t \in Z^+. \tag{2.1.4}$$

The sequence of pxm matrices $W: = W_1, W_2, \ldots$, where $W_i = H\,F^{i-1}\,G$, is called the unit-pulse response sequence of Σ. Furthermore, from (2.1.4) it is seen that the input/output behavior of the system Σ is completely characterized by W. In fact, it is well known that any map which sends sequences $\{u(o), u(1), \ldots\}$ into sequences $\{y(1), y(2), \ldots\}$ and which satisfies "linearity," "causality," and "shift-invariance" conditions, is determined by a matrix sequence $\{W_1, W_2, \ldots\}$ and is defined by the convolution sum $y(t) = \sum_{\ell=0}^{t-1} W_{t-\ell}\,u(\ell)$, $t \in Z^+$.

On the other hand, for continuous time-invariant systems it is well-known (see DESOER [1970], BROCKETT [1970], PADULO and ARBIB [1974]) that the solution, or output response, of (2.1.3) to a zero initial state $x(t_o)$ and an input $u(t)$ on T is given by

$$(2.1.5\text{-a}) \qquad y(t) = \int_{t_o}^{t} W(t - \tau)\, u(\tau)\, d\tau \qquad t \geq t_o,\ t_o,\ t \in T \ ,$$

where

$$(2.1.5\text{-b}) \qquad W(t) = He^{Ft} G \ , \qquad t \geq t_o \ ,$$

is the so called impulse-response matrix associated with Σ. By expanding the right side of (2.1.5-b) into a MacLaurin series (see HO [1966] for details), we have that the i/o map defined by (2.1.5-a) is uniquely determined by the sequence W_i of pxm matrices over R, where $W_i = H F^{i-1} G$ for each $i \in Z^+$.

The above considerations suggest the following abstract definition of the algebraic realization problem for time-invariant systems over an arbitrary field K.

(2.1.6) DEFINITION: A time-invariant input/output (i/o) map f_W is a sequence $\{W_1, W_2, \ldots\}$ of pxm matrices over K. A system $\Sigma = (F,G,H)$ realizes f_W if an only if $W_i = H F^{i-1} G$ for each $i \in Z^+$. A realization $\Sigma = (F,G,H)$ is said to be minimal if its dimension is minimal among all possible realizations of f_W.

Denote by $K^{pxm}[[z^{-1}]]$ the K-linear space of all formal power series over K^{pxm} in the indeterminate z^{-1}. That is, an element W in $K^{pxm}[[z^{-1}]]$ is of the form

$$(2.1.7) \qquad W = \sum_{i=1}^{\infty} W_i z^{-i} , \quad W_i \in K^{pxm} \quad .$$

(2.1.8) DEFINITION: A formal power series $W = \sum_{i=o}^{\infty} W_i z^{-i}$ in $K^{pxm}[[z^{-1}]]$ is said to be recurrent if and only if there exist a positive integer n and elements $\alpha_o, \alpha_1, \ldots, \alpha_{n-1}$ in K such that $W_{n+k} = \sum_{i=o}^{n-1} \alpha_i W_{i+k}$ for all non-negative integers k.

The question of whether or not a time-invariant i/o map $f_W = \{W_1, W_2, \ldots\}$ has a realization is answered next.

(2.1.9) THEOREM. (KALMAN, FALB, and ARBIB [1969]) A time-invariant input/output map f_W is realizable if and only if the formal power series

$$(2.1.10) \qquad W(z^{-1}) := \sum_{i=1}^{\infty} W_i z^{-i} \in K^{pxm}[[z^{-1}]] \quad ,$$

is recurrent in the sense of definition (2.1.8).

Since each W_i is a pxm matrix over K, the formal power series associated with f_W (see (2.1.10)) can also be represented as a matrix $(w_{ij}(z^{-1}))_{pxm}$, where each $w_{ij}(z^{-1})$ is an element of $K[[z^{-1}]]$. A formal power series with scalar coefficients is said to be rational if it is the expansion in z^{-1} of a quotient $p(z)/q(z)$, where $p(z)$, $q(z)$ are polynomials with coefficients in K, and $q(z)$ is a monic polynomial of

degree greater than that of $p(z)$. An arbitrary formal power series $W(z^{-1})$ in $K^{pxm}[[z^{-1}]]$ is rational if all $w_{ij}(z^{-1})$ are rational.

(2.1.10) PROPOSITION: A time-invariant i/o map $f_W = \{W_1, W_2, \ldots\}$ is realizable if and only if its associated power series $W(z^{-1}) = \sum_{i=1}^{\infty} W_i z^{-i}$ is rational.

For a proof of the above see EILENBERG [1974].

In order to give more explicit criteria to decide whether a given f_W is realizable, we formally define the Hankel or behavior matrix of an i/o map f_W as the bi-infinite matrix given by, in block form,

$$B_W = \begin{bmatrix} W_1 & W_2 & W_3 & \cdots \\ W_2 & W_3 & W_4 & \cdots \\ W_3 & W_4 & W_5 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} . \tag{2.1.11}$$

The rank of B_W is defined as the smallest positive integer n such that all minors of B_W of order greater than n are zero. In particular, B_W has infinite rank when there exist non-zero minors of arbitrary order. A well-known criterion of realizability (see Kalman [1968-b]) is given next.

(2.1.12) THEOREM: A time-invariant i/o map $f_W = \{W_1, W_2, \ldots\}$ is realizable if and only if B_W has finite rank n. Furthermore, n is the dimension of a minimal realization.

Next, we review a realization procedure which was developed in SILVERMAN [1971] and for which an abstract interpretation was given by ROUCHALEAU [1972] (see also KALMAN [1976]).

With c_i denoting the i^{th} elementary column of B_W, let X_W^c be the K-linear space consisting of all finite linear combinations $\sum_i \alpha_i c_i$ with $a_i \in K$. Then the condition that rank $B_W = n$ is equivalent to saying that X_W^c has dimension n. Define the following linear map

$$\Phi_\rho: \; X_W^c \longrightarrow X_W^c : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \longrightarrow \begin{bmatrix} x_{1+\rho} \\ x_{2+\rho} \\ x_{3+\rho} \\ \vdots \end{bmatrix} .$$

By definition, Φ_ρ acts on c_i by dropping the first ρ-components of c_i. Now, define

$$\overline{G} : K^m \to X_W^c : \; u \to \sum_{i=1}^{m} u_i \, c_i , \quad u_i = i^{th}\text{-entry of the system input } u$$

$$\overline{H} : X_W^c \to K^p : x \to \overline{H}x = \text{ first } \rho\text{-components of } x \quad .$$

It is not hard to show that the system $\Sigma = (F,G,H)$, where F,G,H are matrix representations of Φ_ρ, $\overline{G}$, and $\overline{H}$, respectively, is a minimal realization of f_W.

The construction of the above realization $\Sigma = (F,G,H)$ was defined in terms of the vector space X_W^c. From the practical point of view, we

would like to give an explicit realization procedure. So, we shall briefly review a very popular algorithm for realization of i/o maps.

Again let B_W denote the infinite Hankel matrix defined by (2.1.11). The condition rank B_W = n is equivalent to the existence of an nxn nonsingular submatrix Φ of B_W. Given Φ, let Γ be the nxm submatrix of the first m elementary columns of B_W whose rows correspond to those of Φ; similarly, let Λ be the pxn submatrix of the first p-elementary rows of B_W whose columns corrospond to those of Φ. Finally, denote by $\sigma\Phi$ the nxn submatrix of B_W whose rows correspond to those of Φ and whose i^{th} column is m-positions to the right of the i^{th} column of Φ. With these conventions, we have the following result.

(2.1.13) THEOREM. (SILVERMAN [1971]). Assume that the time-invariant i/o map $f_W = \{W_1, W_2, \ldots\}$ is realizable, i.e., rank B_W = n, and let Φ be an nxn non-singular submatrix of B_W. Then $(\Phi^{-1}\sigma\Phi, \Phi^{-1}\Gamma, \Lambda)$ is a minimal realization of f_W.

An n-dimensional system $\Sigma = (F,G,H)$ (Σ may represent either a discrete or a continuous system) is said to be reachable if the nxnm reachability matrix $R := [G,FG,\ldots,F^{n-1}G]$ has rank n. Also, we say that Σ is observable if the npxn observability matrix $O^T := [H^T, F^TH^T, \ldots (F^{n-1})^TH^T]^T$ has rank n. Furthermore, Σ is said to be canonical if it is reachable and observable.

For a realizable time-invariant i/o map $f_W = \{W_1, W_2, \ldots\}$ we have the following useful result due to KALMAN [1976].

(2.1.14) PROPOSITION: Let $f_W = \{W_1, W_2, \ldots\}$ be a time-invariant input/output map, and let $\Sigma = (F,G,H)$ be a finite-dimensional realization of f_W. Then Σ is a minimal realization of f_W if only if Σ is canonical.

2.2 Analytic Continuous-Time Systems

Let $t_1, t_2 \in \mathbf{R} \cup \{\infty\}$ such that $t_1 < t_2$. Denote by $A(J)$ the $\mathbf{R}$-linear space of all real-valued analytic functions defined on $J = (t_1, t_2)$. With pointwise addition and multiplication, $A(J)$ becomes a commutative ring with unity $1: J \to \mathbf{R}: t \to 1$. In fact, it follows from SILVERMAN and BUCY [1970], see also KAMEN [1979], that $A(J)$ is a Bezout domain; that is, $A(J)$ is an integral domain and every finitely generated ideal in $A(J)$ is principal.

For any fixed positive integers m,n and p, an m-input p-output n-dimensional linear time-varying system Σ over $A(J)$ is a triple $\Sigma = (F,G,H)$ consisting of nxn, nxm, pxn matrices over $A(J)$, together with the dynamical equations.

$$(2.2.1) \qquad \frac{dx}{dt}(t) = F(t)\, x(t) + G(t)\, u(t) \quad ,$$

$$(2.2.2) \qquad y(t) = H(t)\, x(t) \quad ,$$

where $x(t) \in \mathbf{R}^n$ is the state at time t, $u(t) \in \mathbf{R}^m$ is the input or control at time t, and $y(t) \in \mathbf{R}^p$ is the system output at time t.

Let $C^m(J)$ denote the K-linear space of all m-element column vectors whose components are continuous real-valued functions defined on J. Then as is well known (see KALMAN and WEISS [1965] and KALMAN [1968-a]), the complete solution of (2.2.1) for an initial state $x_1 = x(t_1)$ and for an input $u \in C^m(J)$ is given by

$$(2.2.3) \qquad x(t) = \Phi(t,t_1)x_1 + \int_{t_1}^{t} \Phi(t,\tau)\, G(\tau)\, u(\tau)\, d\tau, \quad t \in J,$$

where $\Phi(t,\tau)$ is the state-transition matrix associated with $F(t)$. From (2.2.2) and (2.2.3), it follows that the output response of Σ due to an input $u \in C^m(J)$ with $x_1 = o$ is given by

$$(2.2.4) \qquad y(t) = \int_{t_1}^{t} W_\Sigma(t,\tau)u(\tau)\, d\tau , \quad t \in J ,$$

where the matrix

$$(2.2.5) \qquad W_\Sigma(t,\tau) = \begin{cases} H(t)\, \Phi(t,\tau)G(\tau) , & t \geq \tau , \\ 0 , & \text{otherwise,} \end{cases}$$

is the so-called impulse-response function matrix. It is easy to see from (2.2.4) that the impulse-response matrix completely characterizes the input/output properties of Σ, which is a causal dynamical system. For many problems of control and estimation it is useful to define $\Phi(t,\tau)$ for $\tau > t$, which leads to the notion of the weighting pattern matrix defined by

(2.2.6) $\bar{W}_\Sigma(t,\tau) = H(t)\Phi(t,\tau)G(\tau)$ for all $t, \tau \in J$.

Since the impulse response $W_\Sigma(t,\tau)$ is analytic, it can be extended uniquely by analytic continuation to the region $\tau > t$. Therefore, for analytic systems (2.2.9), the impulse response and the weighting pattern contain the same information. Since $W_\Sigma(t,\tau) = \bar{W}_\Sigma(t,\tau)$ for all $t, \tau \in J$, we can write

$$(2.2.7) \qquad y(t) = \int_{t_1}^{t} \bar{W}_\Sigma(t,\tau)\, u(\tau)\, d\tau, \quad t \in J.$$

The relation (2.2.7) defines a map $f_\Sigma: C^m(J) \to C^p(J)$: $u \to f_\Sigma(u) = y$ which is usually referred to as the response map (or input/output map) of the system Σ.

For fixed positive integers p, m, let $\bar{W}(t,\tau)$ be a $p \times m$ matrix whose entries are real-valued analytic functions defined on $J \times J$. Let $f_{\bar{W}}: C^m(J) \to C^p(J)$ denote the linear integral operator with weighting pattern (or kernel) $\bar{W}(t,\tau)$; that is,

$$(2.2.8) \qquad [f_{\bar{W}}(u)](t) = \int_{t_1}^{t} \bar{W}(t,\tau)\, u(\tau)\, d\tau, \quad t \in J \text{ and } u \in C^m(J).$$

A system $\Sigma = (F,G,H)$ over $A(J)$ is said to be a realization of the operator $f_{\bar{W}}$ defined by (2.2.8) iff $f_{\bar{W}} = f_\Sigma$, where f_Σ is the response map of Σ given by (2.2.7). A realization of $f_{\bar{W}}$ is minimal if its dimension is minimal among all possible realizations.

Next, we review some realizability criteria for an integral operator $f_{\bar{W}}$ with kernel $\bar{W}(t,\tau)$. We shall start by reviewing a well known result due to SILVERMAN and MEADOWS [1966].

Given $f_{\bar{W}}: C^m(J) \to C^p(J): u \to f_{\bar{W}}(u): t \to \int_{t_1}^{t} \bar{W}(t,\tau)u(\tau)d\tau$, for each positive integer i define

$$(2.2.9)\quad B_{\bar{W}}^{(i)}(t,\tau) = \begin{bmatrix} \bar{W}_{11}(t,\tau) & \bar{W}_{12}(t,\tau) & \dots & \bar{W}_{1i}(t,\tau) \\ \bar{W}_{21}(t,\tau) & \bar{W}_{22}(t,\tau) & \dots & \bar{W}_{2i}(t,\tau) \\ \vdots & & & \\ \bar{W}_{i1}(t,\tau) & \bar{W}_{i2}(t,\tau) & \dots & \bar{W}_{ii}(t,\tau) \end{bmatrix}, \quad t,\tau \in J,$$

where

$$(2.2.10)\quad \bar{W}_{ij}(t,\tau) := \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial^{j-1}}{\partial \tau^{j-1}} \bar{W}(t,\tau) \quad ; i, j \in Z^+ \text{ and } (t,\tau) \in JxJ.$$

Note that the matrices $\bar{W}_{ij}$'s are well defined because of the analyticity of $\bar{W}(\cdot,\cdot)$.

(2.2.11) THEOREM (SILVERMAN and MEADOWS [1966]). *An integral operator $f_{\bar{W}}$ is realizable if and only if there exists a finite positive integer n such that $B_{\bar{W}}^{(n)}(t,\tau)$ and $B_{\bar{W}}^{(n+1)}(t,\tau)$ both have rank n almost everywhere for $t \geq \tau$. Furthermore, n is the dimension of a minimal realization of $f_{\bar{W}}$.*

It is worthwhile to observe that the above criterion consists basically in checking the rank of an infinite collection of finite matrices indexed by $t, \tau \in J$ with $t \geq \tau$.

KALMAN [1968a] derived a necessary and sufficient condition for realizability of a response map $f_{\bar{W}}$ specified in terms of the rank of the matrices $B^{i}_{\bar{W}}(t, \tau)|_{\tau=t}$, $t \in J$.

(2.2.12) THEOREM (KALMAN [1968b]). An integral operator $f_{\bar{W}}: C^{m}(J) \to C^{p}(J)$ is realizable if, and only if, there is a positive integer n, and for every closed bounded interval $I \subset J$ there is a positive integer n_I such that rank $B^{(k)}_{\bar{W}}(t, t) = n$ for all $t \in I$, and all $k \geq n_I$. Further, n is the dimension of a minimal realization.

Even though the above result may be considered as a one-variable criteria, i.e., given in terms of $B^{(i)}_{\bar{W}}(t, t)$ instead of $B^{(i)}_{\bar{W}}(t, t)$, it has the disadvantage that it requires the computation of the rank of an infinite matrix at each $t \in J$.

It was in FLIESS [1982, 1983] that a simpler criteria (at least from theoretical point of view) for realizability of real analytic i/o maps was developed. His work is based on the theory of representative series (see Chapter Three) and on the theory of germs of analytic functions.

It is well-known that the kernel $\bar{W}(t,\tau)$ of any analytic linear operator $f_{\bar{W}}: C^{m}(J) \to C^{p}(J)$ has a "natural" power series, its Taylor series expansion; that is, at each $(t, \tau) \in J \times J$

$$(2.2.13)\quad \bar{W}(t,\tau) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{W}_{ij} t^{i-1}\tau^{j-1} \in R^{pxm}[[t, \tau]] \quad ,$$

where

$$(2.2.14)\quad \bar{W}_{ij} = \frac{1}{(i-1)!(j-1)!}\left[\frac{\partial^{i-1}}{\partial t^{i-1}}\frac{\partial^{j-1}}{\partial \tau^{j-1}}\bar{W}(t, \tau)\right]\Big|_{t=\tau=0} \text{ for each } i,j\in Z^{+}.$$

(assume that $0 \in J$).

For an operator $f_{\bar{W}}$ with kernel $\bar{W}(t, \tau)$ given by (2.2.13), define as its characteristic matrix $C(f_{\bar{W}})$ the following infinite block matrix

$$(2.2.15)\quad C(f_{\bar{W}}) := \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} & \bar{W}_{13} & \cdots \\ \bar{W}_{21} & \bar{W}_{22} & \bar{W}_{23} & \cdots \\ \bar{W}_{31} & \bar{W}_{32} & \bar{W}_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} ,$$

where the $\bar{W}_{ij}$'s are given by (2.2.14).

Note that $C(f_{\bar{W}})$ is not a Hankel matrix as in (2.1.11) because it is not even block symmetric. With the usual definition of rank for an infinite matrix we have the next result.

(2.2.16) THEOREM: (FLIESS [1982]). A scalar analytic i/o map $f_{\bar{W}}$: $C(J) \to C(J)$ with kernel $\bar{W}(t, \tau)$ is realizable if and only if there exists a finite non-negative integer n such that rank $C(f_{\bar{W}}) = n$. Furthermore, n is the dimension of a minimal realization.

Hence, to check for realizability of a scalar $f_{\overline{W}}$, we have to calculate the rank of a single infinite matrix. This fact "reduces" the computations considerably. Unfortunately, Fliess does not consider either a theory of representative series with matrix coefficients or a concrete minimal realization of the scalar i/o map $f_{\overline{W}}$. However, from the results of Chapter Three in this work, the multiple input/multiple output version of Theorem (2.2.16) can be deduced.

Next we review a realization procedure for real analytic i/o maps that was developed in KAMEN [1979] and which can be considered as a generalization of the Silverman-Meadows algorithm (see SILVERMAN [1971]).

Given a linear operator $f_{\overline{W}}$, specified by

$$f_{\overline{W}}: C^m(J) \to C^p(J): u \to f_{\overline{W}}(u) = y: t \to \int_{t_1}^{t} \overline{W}(t, \tau)u(\tau)d\tau \quad ,$$

for each $t \in J$ define the infinite matrix

$$B_{\overline{W}}(t) = \begin{bmatrix} \overline{W}_{11}(t) & \overline{W}_{12}(t) & \overline{W}_{13}(t) & \cdots \\ \overline{W}_{21}(t) & \overline{W}_{22}(t) & \overline{W}_{23}(t) & \cdots \\ \overline{W}_{31}(t) & \overline{W}_{32}(t) & \overline{W}_{33}(t) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad , \tag{2.2.17}$$

where

$$\overline{W}_{ij}(t) = \frac{\partial^{i-1}}{\partial t^{i-1}} \frac{\partial^{j-1}}{\partial \tau^{j-1}} \overline{W}(t, \tau)\Big|_{\tau=t} \quad , \quad i,j \in Z^+ \quad .$$

The matrix $B_{\underline{w}}(t)$ is called the (generalized) Hankel matrix associated with $f_{\underline{w}}$ at time t. Note also that since the entries of $B_{\underline{w}}$ are elements of A(J), $B_{\underline{w}}$ can be seen as an infinite matrix over the ring A(J).

Let $r_i(c_i)$ denote the i^{th} elementary row (column) of the Hankel matrix $B_{\underline{w}}$. Let $X_{\underline{w}}^r(X_{\underline{w}}^c)$ denote the A(J)-module generated by the $r_i(c_i)$. Here addition and scalar multiplication are defined componentwise. Then it is not hard to see that a) since A(J) is a commutative ring, $X_{\underline{w}}^r$ and $X_{\underline{w}}^c$ can be defined as left or right modules over A(J), b) $X_{\underline{w}}^r$ and $X_{\underline{w}}^c$ are torsion-free modules, i.e., if ax = 0 for some $a \in A(J)$, $x \in X_{\underline{w}}^r$ or $x \in X_{\underline{w}}^c$, then a must be zero.

Finally denote by $B_{\underline{w}}^{(i)}(t)$ the submatrix of $B_{\underline{w}}(t)$ given by

$$B_{\underline{w}}^{(i)}(t) = \begin{bmatrix} \overline{w}_{11}(t) & \overline{w}_{12}(t) & \dots & \overline{w}_{1i}(t) \\ \overline{w}_{21}(t) & \overline{w}_{22}(t) & \dots & \overline{w}_{2i}(t) \\ \vdots & & & \\ \overline{w}_{i1}(t) & \overline{w}_{i2}(t) & \dots & \overline{w}_{ii}(t) \end{bmatrix}, \quad t \in J.$$

(2.2.18) THEOREM (KAMEN [1979]): *Given the operator $f_{\underline{w}}$ with associated Hankel matrix $B_{\underline{w}}$, the following are equivalent.*

(i) *The operator $f_{\underline{w}}$ is realizable and the dimension of a minimal realization is n.*

(ii) *The modules $X_{\underline{w}}^c$ and $X_{\underline{w}}^r$ are finite and free with* rank $X_{\underline{w}}^c$ = rank $X_{\underline{w}}^r$ = r.

(iii) rank $B_{\underline{w}}^{(i)}(t) = n$ *for all* $k > \mu$, *all* $t \in J$, *for some integer* μ.

Kamen has shown the above theorem by giving a concrete realization of $f_{\bar{w}}$. A key contruct in his algorithm is the definition of the pseudolinear differential transformation

$$(2.2.19)\qquad \lambda_p : X_{\bar{w}}^c \to X_{\bar{w}}^c : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \to \begin{bmatrix} x_{1+p} \\ x_{2+p} \\ x_{3+p} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} + \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

where p is a fixed positive integer.

To see that λ_p is well-defined and what properties it has. the reader is referred to KAMEN [1979].

Let us assume that $X_{\bar{w}}^c$ is finite and free with rank n, and define the following A(J)-linear maps

$$\bar{G} : A^m(J) \to X_{\bar{w}}^c : \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \to \sum_{i=1}^{m} u_i c_i \qquad c_i = i^{th}\text{-elementary column of } B_{\bar{w}},$$

(2.2.20)

$$\text{(2.2.21)} \qquad \bar{H}\colon X_{\bar{W}}^{c} \to A^{p}(J)\colon \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} .$$

Then it can be shown (see KAMEN [1979]) that a matrix representation for $(\lambda_p, \bar{G}, \bar{H})$ yields a minimal realization of $f_{\bar{W}}$ of dimension n.

To compute a minimal realization of $f_{\bar{W}}$, the following is needed.

KAMEN'S ALGORITHM FOR CONSTRUCTION OF MINIMAL REALIZATIONS

(i) Find a positive integer μ such that rank $B_{\bar{W}}^{(i)}(t) = n$ for all $i \geq \mu$, all $t \in J$.

(ii) Let N(t) be any sxq submatrix of $B_{\bar{W}}^{(\mu)}$ such that $s \geq p$ and

$$B_{\bar{W}}^{(\mu)}(t) = \left[\begin{array}{c|c|c} \cdot & \cdot & \cdot \\ \hline \vdots & N(t) & \vdots \\ \hline \cdot & \cdot & \cdot \end{array}\right] ,$$

with rank N(t) = n for all $t \in J$.

(iii) Find matrices E(t) and Q(t) such that $N(t)E(t) = [Q(t) \vdots 0]$. It is important to emphasize that E and Q can be found via a finite sequence of multiplications on the matrix N using qxq invertible matrices over A(J).

(iv) Let Q(t) denote the submatrix of the first block column of $B_{\bar{W}}^{(\mu)}$ corresponding to the rows of N. Let σN be the submatrix of $B_{\bar{W}}^{(\mu+1)}$ whose first column is located m

elementary columns to the right of the first column of N and whose rows correspond to those of N.

(v) Define

$$F := \text{first } n\text{-columns of } -(Q^TQ)^{-1}Q^T((\sigma N)E + N\dot{E}) \quad (2.2.20)$$

$$G := [Q^TQ]^{-1}\, Q^T\Omega \quad (2.2.21)$$

$$H := \text{first } p\text{-rows of } Q \quad (2.2.22)$$

Then $\Sigma = (F, G, H)$ is a minimal realization of $f_{\mathfrak{M}}$.

It is worthwhile to emphasize that F, G, H are the matrix representations of λ_p, $\tilde{G}$, $\tilde{H}$ (see (2.2.19-21)) with respect to some basis $v_1, v_2, \ldots, v_n$ of $X^c_{\mathfrak{M}}$ satisfying

$$[v_1(t), v_2(t), \ldots, v_n(t)] = \begin{bmatrix} Q(t) \\ \vdots \end{bmatrix} :$$

and with respect to the respective standard bases of $A^m(J)$ and $A^p(J)$.

Let $A(J)^{nxm}$ denote the set of nxm matrices over A(J). Given the m-input p-output n-dimensional system $\Sigma = (F, G, H)$ over A(J), define

$$D_F : A(J)^{nxm} \to A(J)^{nxm}: \quad M \to -FM + \dot{M} \quad (2.2.23)$$

$$D_F^{\wedge} : A(J)^{nxp} \to A(J)^{nxp} : M \to -\hat{F}M + \dot{M} \quad (2.2.24)$$

where $\dot{M}$ is the derivative of M and $\hat{F} = -F^T$. The maps D_F and $D_{\hat{F}}$ are called the differential transformations associated with Σ.

For each positive integer i, let R_i and O_i denote, respectively, the i^{th} reachability and i^{th}-observability matrix of Σ; that is

$$R_i = [G, D_F(G), \ldots, D_F^{i-1}(G)] , \tag{2.2.25}$$

$$O_i = \begin{bmatrix} (H^T)^T \\ (D_{\hat{F}}(H^T))^T \\ \vdots \\ (D_{\hat{F}}^{i-1}(H^T))^T \end{bmatrix} , \tag{2.2.26}$$

where $D_F^{i-1}(D_{\hat{F}}^{i-1})$ is the (i - 1) - fold composition of $D_F(D_{\hat{F}})$.

A system $\Sigma = (F, G, H)$ over A(J) is said to be reachable (observable) iff rank $R_n(t) = n$ (rank $O_n(t) = n$) for each $t \in J$. The system Σ is canonical if it is both reachable and observable.

(2.2.27) DEFINITION: A continuous time-varying system $\Sigma = (F, G, H)$ is said to be a constant rank system if there exist positive integers α, β, n_r and n_o such that F(t), G(t), and H(t) are max (α, β)-1, α and β times continuously differentiable, and such that

$$\text{rank } R_\alpha(t) = \text{rank } R_{\alpha+1}(t) = n_r \leq n \quad \text{for each } t \in \mathbf{R}$$

$$\text{rank } O_\beta(t) = \text{rank } O_{\beta+1}(t) = n_o \leq n \quad \text{for each } t \in \mathbf{R} \quad .$$

From results due to Silverman [1971], we have the following useful fact.

(2.2.28) THEOREM. Let $f_{\tilde{W}} : C^m(J) \to C^b(J)$ be an integral operator with kernal $\tilde{W}(t, \tau)$ over $A(JxJ)$, and let Σ be a constant-rank realization of $f_{\tilde{W}}$. Then Σ is a minimal realization if and only if it is canonical.

As noted by SILVERMAN [1971] when J is bounded any analytic system Σ is constant rank. In KAMEN [1979], it was shown that when J is unbounded an analytic system is not in general constant rank. Furthermore, he has shown the following interesting result which we shall use later for comparison purposes.

(2.2.29) THEOREM (KAMEN [1979]). Let $f_{\tilde{W}} : C^m(J) \to C^p(J)$ be an operator in the sense of (2.2.8). Then $f_{\tilde{W}}$ is realizable by a constant-rank system if and only if there exist positive integers n and μ such that

$$\text{rank} B_{\tilde{W}}^{(i)}(t) = n \quad \text{for all } i \geq \mu \text{ and all } t \in J \quad .$$

In addition, n is the dimension of a minimal realization.

2.3 Discrete Time-Varying Systems

Let Z denote the ring of integers and let K be an arbitrary, but otherwise fixed, field. For any n in Z^+, the set of all positive integers, we let K^n denote the K-vector space of all n-element column vectors over the field K. Let B be the K-linear space of all functions with domain Z and codomain K. With addition and multiplication defined by

$$(a + b)(k) = a(k) + b(k)$$

$$(ab)(k) = a(k)b(k)$$

for all a, b in B, and all $k \in Z$, B becomes a commutative ring with identity 1, where 1(k) = 1 for all $k \in Z$.

Let $\sigma : B \to B : a \to \sigma a : k \to a(k - 1)$ be the right-shift operator on B. The operator σ is a (ring) automorphism on B; that is, σ is bijective and $\sigma(a + b) = \sigma a + \sigma b$, $\sigma(ab) = (\sigma a)(\sigma b)$ for all $a,b \in B$. The inverse σ^{-1} of σ is the left-shift operator on B defined by $\sigma^{-1}: B \to B \cdot \quad a \to \sigma^{-1}a : k \to a(k+1)$. The ring B with the automorphism σ is said to be a difference ring.

Let $B_+(B_-)$ denote the subring of B consisting of all functions with support bounded on the left (right); that is, for any $a \in B_+$ (B_-) there exists an integer k_a, which depends on a in general, such that a(k) = 0 for all $k < k_a$ ($k > k_a$). For any positive integer n, let B_+^n (B_-^n) denote

the K-vector space of all n-element column vectors with entries in B_+ (B_-).

A subring A of B is said to be a difference subring if A is closed under σ, i.e., $\sigma(A) = A$. In terms of these concepts we have the following notion of a system.

(2.3.1) DEFINITION: Let m,n and p be fixed positive integers and A be a difference subring of R. An m-input p-output n-dimensional linear time-varying discrete-time system over A is a triple $\Sigma = (F, G, H)$ of nxn, nxm, pxn matrices over A, together with the dynamical equations

$$x(k + 1) = F(k)x(k) + G(k)u(k) \tag{2.3.2}$$

$$y(k) = H(k)x(k) \tag{2.3.3}$$

, $k \in Z$.

where $x(k) \in K^n$ is the state of the system at time $k \in Z$, $u(k) \in K^m$ is the system input or control law at time k, and $y(k) \in K^p$ is the system output at time k.

As in KAMEN and HAFEZ [1979], we shall work with a modified version of the standard equations (2.3.2) and (2.3.3). Apply the right-shift operator σ to both sides of (2.3.2) and let $D(k) := F(k - 1)$, $E(k) := G(k - 1)$. This gives the following equation

$$x(k) = D(k)x(k - 1) + E(k)u(k - 1) \quad . \tag{2.3.4}$$

Henceforth, we shall be specifically concerned with the time-varying system $\Sigma = (D, E, H)$ over A defined by the dynamical equations (2.3.4) and (2.3.3).

The state transition matrix of a system Σ is the nxn matrix function $\phi_\Sigma(k, l)$ defined by

$$\phi_\Sigma(k, l) = \begin{cases} D(k)D(k-1) \ldots D(l+1) , & k > l , \\ I , & k = l , \\ \text{undefined} & k < l , \end{cases} \tag{2.3.5}$$

where I is the nxn identity matrix.

As is well known, the unit-pulse response function $W_\Sigma(k, l)$ is given by

$$W_\Sigma(k, l) = \begin{cases} H(k)\phi(k, l+1) E(l+1), & k > l , \\ 0_{pxm} , & k = l , \\ \text{undefined for } k < l , \end{cases} \tag{2.3.6}$$

where 0_{pxm} is the zero matrix of dimension pxm.

The input/output behavior of a system Σ is described by its input/output map $f_\Sigma: B_+^m \to B_+^p$ given by

$$[f_\Sigma(u)](k) = \sum_{l=-\infty}^{k} W_\Sigma(k, l)u(l) \quad , \quad u \in B_+^m \ , \tag{2.3.7}$$

(2.3.8) DEFINITION: Given a linear system $\Sigma = (D, E,, H)$ over A, its dual is a system $\hat{\Sigma} = (\hat{D}, \hat{E}, \hat{H})$ over A, where

$$(2.3.9)\qquad \begin{cases} \hat{D} = \sigma^{-1}D^T \\ \hat{E} = H^T \\ \hat{H} = \sigma^{-1}E^T \end{cases} \qquad (T = \text{transpose operation})$$

and whose dynamical equations are

$$(2.3.10)\qquad z(k-1) = \hat{D}(k)z(k) + \hat{E}(k)v(k) \quad ,$$

$$(2.3.11)\qquad w(k) = \hat{H}(k)z(k) \quad , k \in Z \quad .$$

where $z(k) \in K^n$ is the system state at time, k, $v(k) \in K^p$ is the system input input applied at time k, and $w(k) \in K^m$ is the system response or system output at time k.

It is interesting to observe that $\hat{\Sigma}$ evolves in reverse time.

The state transition matrix of the dual $\hat{\Sigma}$ of a system Σ over A, is defined as

$$(2.3.12)\qquad \hat{\phi}(k, 1) = \begin{cases} D^T(k+2)D^T(k+1) \ldots D^T(1+1) & , k < 1 \\ I & , k = 1 \\ \text{undefined} & , k > 1 \end{cases} ,$$

where I = nxn identity matrix.

It is very easy to show that the unit-pulse response matrix function $W_{\hat{\Sigma}}$ associated with $\hat{\Sigma}$ is given by

$$(2.3.13) \qquad W_{\hat{\Sigma}}(k, l) = \begin{cases} E^T(k+1)\hat{\phi}(k, l-1)H^T(l) , & k < l , \\ 0_{mxp} , & k = l , \\ \text{undefined} , & k > l, \end{cases}$$

where $0_{mxp} = [0]_{mxp}$.

The input/output behavior of $\hat{\Sigma}$ is given by its input/output map

$$(2.3.14) \qquad f_{\hat{\Sigma}}: B_-^p \to B_-^m : v \to f_{\hat{\Sigma}} : k \to \sum_{l=k}^{+\infty} W_{\hat{\Sigma}}(k, l)v(l) \quad ,$$

where $f_{\hat{\Sigma}}$ will be referred to as the response map of the dual system $\hat{\Sigma}$.

By comparing (2.3.5) and (2.3.12) we observe at once that

$$(2.3.15) \qquad \hat{\phi}(k, l-1) = \phi^T(l, k+1) \quad \text{for all} \quad k < l \quad ,$$

and we get the following result

(2.3.16) PROPOSITION: Given a linear system $\Sigma = (D, E, H)$ over $\mathbf{A}$, its unit pulse response matrix function W_Σ is related to the corresponding one $W_{\hat{\Sigma}}(\cdot, \cdot)$ of the dual system $\hat{\Sigma}$ through the relation

$$(2.3.17) \qquad W_\Sigma(k,l) = W_{\hat{\Sigma}}^T(l, k) \quad \text{for all } k > l \quad .$$

PROOF: Take the transpose of $W_\Sigma^\wedge(\cdot, \cdot)$ in (2.3.13), use (2.3.15), and compare with (2.3.6). □

(2.3.18) DEFINITION: Let m and p be fixed positive integers. An m-input p-output linear time-varying causal input/output map f_W is a K-linear map

$$f_W : B_+^m \to B_+^p ,$$

such that if $u(k) = 0$, $k < k_u$, for some $u \in B_+^m$, then $[f_W(u)](k) = 0$ for all $k \leq k_u$.

It is a well known fact that for any input/output map f_W as defined above, there exists a pxm matrix function $W(k, l)$ such that for any $u \in B_+^m$,

$$(2.3.19) \qquad [f_W(u)](k) = \sum_{l=-\infty}^{k} W(k, l)u(l) \quad .$$

By causality $W(k, l) = 0$ for $k < l$. The matrix function W is the unit-pulse response function associated with the input/output map f_W. Furthermore, if we denote by CP the (strictly) causal half-plane of ZxZ; that is,

$$(2.3.20) \qquad CP = \{(k, l) \in Z \times Z : k > l\}$$

we observe that W is a function with domain $CP \cup \{(k, l) \in Z \times Z : k = l\}$ and codomain $K^{p \times m}$ satisfying $W(k, k) = 0$ for all $k \in Z$.

Our next concept is the notion of realizability.

(2.3.21) DEFINITION: Let $\Sigma = (D, E, H)$ be an m-input p-output n-dimensional linear system over A and $f_W : B_+^m \to B_+^p$ be an i/o map defined by (2.3.19). Σ is said to be a realization of f_W if, and only if, $f_W = f_\Sigma$, where f_Σ is the response map of Σ given by (2.7). The i/o map f_W is said to be realizable if there exists a realization. A realization is minimal if there is no realization having smaller dimension.

Assume that $\Sigma = (D, E, H)$ is a realization over A of f_W, then from the above definition

$$f_W(u) = f_\Sigma(u) \quad , \quad u \in B_+^m \quad .$$

That is

$$\sum_{l=-\infty}^{k} [W(k, l) - W_\Sigma(k, l)]u(l) = 0 \quad , \quad u \in B_+^m, \; k \in Z \; .$$

But it is not hard to see that such an equality implies

(2.3.22) $$W(k, l) = W_\Sigma(k, l) \quad \text{for all} \quad (k, l) \in CP \quad .$$

Conversely, if (2.3.22) holds, then it must be clear that Σ realizes f_W. We have thus shown

(2.3.23) PROPOSITION: A system $\Sigma = (D, E, H)$ over A is a realization of the i/o map f_W if, and only if

$$W(k, l) = W_\Sigma(k, l) \quad \text{for all} \quad (k, l) \in CP \quad .$$

Given a linear i/o map $f_W : B_+^m \to B_+^p$ with unit-pulse response function $W(\cdot, \cdot)$, define as its dual, $f_{\hat{W}}$, the linear i/o map given by

$$(2.3.24) \qquad f_{\hat{W}} : B_-^p \to B_-^m : v \to f_{\hat{W}}(v) : k \to \sum_{l=k}^{\infty} \hat{W}(k, l)v(l)$$

where

$$\hat{W}(k, l) = W^T(l, k) \quad \text{for all} \quad (l, k) \in CP \quad .$$

The above concept leads us to the following result.

(2.3.26) PROPOSITION: If a given i/o map $f_W : B_1^m \to B_+^p$ with unit-pulse response $W(\cdot, \cdot)$ is realizable by the system $\Sigma = (D, E, H)$ over A, then its dual map $f_{\hat{W}} : B_-^p \to B_-^m$ is realizable by the system $\hat{\Sigma} = (\sigma^{-1}D^T, H^T, \sigma^{-1}E^T)$, the dual of Σ. Conversely, if the system $\Sigma = (D, E, H)$ over A is a realization of $f_{\hat{W}}$, then the dual system $\hat{\Sigma}$ is a realization of f_W.

In definitions (2.3.1) and (2.3.21), it is important to note that by selecting the difference subring A, we can restrict attention to a particular class of systems. For instance, if $A = l^{\infty}(Z)$ = set of all bounded functions from Z into R, we are able to study linear time-varying systems with bounded coefficients. However, in this work we shall assume, in general, that A = B, unless, of course, we need to make more specific any result.

Next, we review some basic definitions and some structural properties of linear n-dimensional discrete time-varying systems which will be used in the subsequent work.

We shall start by introducing the concept of a semilinear transformation (s.l.t.) associated to a system $\Sigma = (D, E, H)$ over B (see definition (2.3.1)) which was used for the first time by KAMEN and HAFEZ [1979] in the study of discrete time-varying systems.

Let B^n represent the free B-module consisting of all n-element column vectors over B. Given an nxn matrix M over B, let S denote the operator defined by

$$(2.3.27) \qquad S : B^n \to B^n : v \to M(\sigma v)$$

where $(\sigma v)(k) = v(k - 1)$, $k \in Z$.

By using (2.3.27) we observe that S has the following properties:

(i) S is additive; that is, $S(v_1 + v_2) = S(v_1) + S(v_2)$ for all $v_1, v_2 \in B^n$. To see that, note

$$S(v_1 + v_2) = M\sigma(v_1 + v_2) = M(\sigma v_1 + \sigma v_2) = M\sigma v_1 + M\sigma v_2$$

$$= S(v_1) + S(v_2).$$

(ii) S is not homogeneous. For let $a \in B$, and let $v \in B^n$, then we have that

$$S(av) = M(\sigma(av)) = M(\sigma a)(\sigma v) = (\sigma a)M(\sigma v) = (\sigma a)S(v) \neq aS(v).$$

Therefore, S is not a linear operator with respect to the B-module structure on B^n. The operator S is said to be a semilinear transformation relative to σ (see JACOBSON [1937]).

Next, let $\Sigma = (D, E, H)$ be an n-dimensional system over B (see definition (2.3.1)). The operator $S : B^n \to B^n : v \to D(\sigma v)$ will be referred to as the semilinear transformation (s.l.t.) of the system $\Sigma = (D, E, H)$. Also, let us define the following recurrent relation:

$$S^i E := D[\sigma(S^{i-1}E)] \quad , \quad i \in Z^+$$

(2.3.28)

$$S^0 = I = \text{identity operator} \quad .$$

We shall also find useful the notion of the dual of a s.l.t. S (see HAFEZ and KAMEN [1979]).

(2.3.29) DEFINITION: Consider a given s.l.t. S (see (2.3.27)), its dual $\hat{S}$ is the mapping $\hat{S}$ defined by

$$(2.3.30) \qquad \hat{S} : B^n \to B^n : v \to \sigma^{-1}(M^T v)$$

where M is the same nxn matrix over B used to define S.

It is not hard to show that the dual $\hat{S}$ of a s.l.t. S relative to σ is a s.l.t. relative to σ^{-1}.

Next, consider the dual system $\hat{\Sigma} = (\hat{D}, \hat{E}, \hat{H})$ over B of $\Sigma = (D, E, H)$ (see definition (2.3.8)) whose dynamical equations are

$$z(k - 1) = D^T(k + 1)z(k) + H^T(k)v(k) \quad ,$$

$$w(k) = E^T(k + 1)z(k) \quad .$$

The operator $\hat{S} : B^n \to B^n : \alpha \to \sigma^{-1}(D^T\alpha)$ will be referred to as the s.l.t. of the dual system $\hat{\Sigma}$. Also, let us define the following recursive expression

$$(2.3.31) \qquad \hat{S}^j H^T = \sigma^{-1}[D^T(\hat{S}^{j-1}H^T)] \quad , \quad j \in Z^+$$
$$\hat{S}^0 = I \ = \text{identity operator.}$$

A system $\Sigma = (D, E, H)$ over B is said to be reachable (in N steps) at all times if and only if there exists a positive integer N such that for any $k \in Z$ and any $x \in B^n$ there exists an input sequence

$u(k - N)$, $u(k - N + 1)$, ..., $u(k - 1)$ which drives Σ from $x(k - N) = 0$ to $x(k) = x$. The dual notion of observability in N steps can be given the usual system interpretation. That is, Σ is observable (in N steps) if its dual $\hat{\Sigma}$ is reachable in N steps.

It is possible to find necessary and sufficient conditions for a given system $\Sigma = (D, E, H)$ to be either reachable or observable in N steps by using the system's semilinear transformations S and $\hat{S}$. The following results are basically due to WEISS [1972] (see also KAMEN and HAFEZ [1977]).

For any fixed positive integer i, denote by $\mathbf{R}_i(\mathbf{O}_i)$ the i^{th} reachability (observability) matrix of Σ defined by

$$\mathbf{R}_i = [E, SE, \ldots, S^{i-1}E] \quad , \tag{2.3.32}$$

$$\mathbf{O}_i^T = [H^T, \hat{S}H^T, \ldots, \hat{S}^{i-1}H^T]) \quad .$$

(2.3.33) THEOREM: An n-dimensional system $\Sigma = (D, E, H)$ over $\mathcal{B}$ is

(i) reachable in N steps at all time if, and only if

$$\text{rank } \mathbf{R}_N(k) = n \quad , \text{ for all } k \in Z \quad ,$$

i.e., if and only if R_N is right-invertible over $\mathcal{B}$.

(ii) Observable in N steps at all times if, and only if

$$\text{rank } O_N(k) = n \quad , \quad \text{for all } k \in Z$$

i.e. if and only if O_N is left-invertible over B.

A system $\Sigma = (D, E, H)$ over B is said to be canonical if it is both reachable and observable.

(2.3.34) REMARK: Here we observe another basic difference between continuous-time and discrete-time systems. That is, the notion of reachability is not the same for both of them. For the latter kind of systems, reachability (in N steps) is expressed in terms of a rank condition on the N^{th}-reachability matrix $R_N := [E, \ldots, S^{N-1}E]$ at each $k \in Z$, often $N > \dim \Sigma$ (see example (2.3.35)); furthermore it has a definite dynamical interpretation. While, for continuous-time system the natural counterpart would be to define reachability in terms of the rank of $R_N := [G, D_F G, \ldots D_F^{N-1}G]$ at each $t \in J$ with $N \geq \dim \Sigma$; however, such a notion has not been used nor has it been given a system interpretation. By duality, the above also holds for observability.

Such a discrepancy is another reason why it is not true that the theory of realization for discrete-time systems is just a carry over of the existing one for continuous-time systems.

(2.3.35) EXAMPLE: Suppose that $D(k) = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, $e(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By using (2.3.28) it is not hard to see that

$$(Se)(k) = \begin{bmatrix} k \\ 1 \end{bmatrix}, \quad (S^2e)(k) = \begin{bmatrix} 2k-1 \\ k \end{bmatrix} \quad .$$

Hence,

$$R_2(k) = [e(k)(Se)(k)] = \begin{bmatrix} 0 & k \\ 1 & 1 \end{bmatrix} ,$$

clearly

$$\text{rank } R_2(k) = \begin{cases} 1 \quad , & k = 0 \\ 2 \quad , & k \neq 0 \end{cases} .$$

On the other hand,

$$\text{rank } R_3(k) = \text{rank} \begin{bmatrix} 0 & k & 2k-1 \\ 1 & 1 & k \end{bmatrix} = 2 \quad \text{for all } k \in Z \quad .$$

That is, $\Sigma = (D, e, -)$ is reachable in three steps; i.e., $N = 3$; however, it is not reachable in two steps (the dimension of Σ).

The first work on realization theory of discrete time-varying systems is due to EVANS [1972] and WEISS [1972]. We shall review first some results of the first author.

Recall that the causal half-plane CP has been defined as

$$CP = \{(k, l) \in Z \times Z : k > l\}$$

(see Figure 1).

Let λ be any fixed integer, and let us denote by CP_λ the following proper subset of CP,

$$CP_\lambda = \{(k, l) \in Z \times Z : k \geq \lambda > l\} \quad .$$

It is immediate to observe that the family $\{CP_\lambda : \lambda \in Z\}$ has the next properties,

(i) Each CP_λ is rectangular in form (see Figure 2);

(ii) $\{CP_\lambda : \lambda \in Z\}$ forms a cover for CP; that is,

$$CP = \bigcup_{\lambda \in Z} CP_\lambda \quad .$$

(2.3.37) DEFINITION: A system $\Sigma = (D, E, H)$ over B is said to realize a linear i/o map $f_W : B_+^m \to B_+^p$ on CP_λ, λ = fixed integer, if and only if

(2.3.38) $W(k, l) = W_\Sigma(k, l)$ for all $(k, l) \in CP_\lambda$,

where $W_\Sigma(\cdot, \cdot)$ is the unit-pulse response matrix of Σ (see (2.3.6)). In such a case, we also say that f_W is realizable on CP_λ.

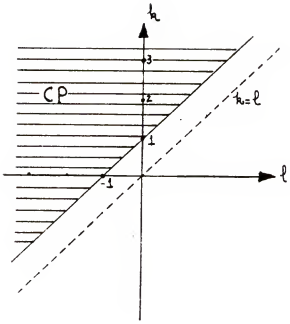


Figure 1

Causal Half-Plane
Corresponds to Hatched Region.

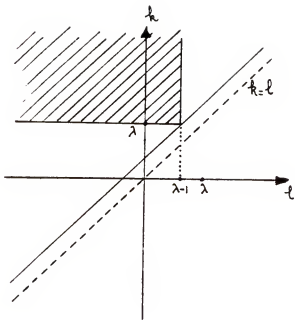


Figure 2

CP_λ
Corresponds to Hatched Region

(2.3.38) PROPOSITION (EVANS [1972]): An input/output map $f_W: B_+^m \to B_+^p$ with unit pulse response matrix $W(\cdot, \cdot)$ is realizable on CP_λ, where λ is a fixed integer, if and only if there exist matrix functions

(2.3.39) $$P_\lambda(\cdot) : \{k \in Z : k > \lambda\} \to K^{p \times n} \quad ,$$

(2.3.40) $$Q_\lambda(\cdot) : \{l \in Z : \lambda > l\} \to K^{n \times m} \quad ,$$

with n finite, such that

(2.3.41) $$W(k, l) = P_\lambda(k)Q_\lambda(l) \quad \text{for all } (k,l) \in CP_\lambda \quad .$$

PROOF: (Necessity). Assume that f_W is realizable on CP_λ by a system $\Sigma = (D, E, H)$ of dimension n. Then by definition

$$W(k, l) = H(k)\phi(k, l + 1)E(l + 1) \quad , \text{ for all } (k, l) \in CP_\lambda .$$

But

$$\phi(k, l + 1) = \phi(k, \lambda + 1)\phi(\lambda, l + 1) \quad , \text{ for all } (k, l) \in CP_\lambda .$$

Thus

$$W(k, l) = P_\lambda(k)Q_\lambda(l) \quad \text{for} \quad (k, l) \in CP_\lambda$$

where

$$P_\lambda(k) := H(k)\phi(k, \lambda + 1) \quad , \quad k \geq \lambda$$

$$Q_\lambda(l) := \phi(\lambda, l + 1)E(l + 1) \quad , \quad \lambda > l \quad .$$

It must be clear that P_λ and Q_λ satisfy the requirements of the proposition

(Sufficiency). Suppose that there exist pxn and nxm matrices $P_\lambda(\cdot)$ and $Q_\lambda(\cdot)$ satisfying (2.3.39) and (2.3.40) together with (2.3.41). Then, it is obvious that the system $\Sigma = (I_n, E, H)$ is an n-dimensional realization of f_W on CP_λ, where $E(k) = Q_\lambda(k-1)$, for $k \leq \lambda$ and $E(k) = 0$ for $k > \lambda$; $H(k) = P\lambda(k)$ for $k \geq \lambda$ and $H(k) = 0$ for $k < \lambda$.

□

The next result links the individual realizations of f_W on each CP_λ to get a criteria for realizability on the entire causal half-plane CP.

(2.3.42) PROPOSITION (EVANS [1972]): An i/o map $f_W : B_+^m \to B_+^p$ is realizable on CP if, and only if, for each $\lambda \in Z$ f_W is realizable on CP_λ by an n-dimensional, linear, time-varying, discrete-time system $\Sigma = (D, E, H)$, for some finite n independent of λ.

The greatest drawback of the above criteria is that it is necessary to check realizability of f_W on each CP_λ by factorizing $W(\cdot, \cdot)$ as in (2.3.41). And it is a well-known fact that the separation of $W(k,l)$ into a product of two matrices $P_\lambda(k)$ and $Q_\lambda(l)$ is not an easy task. It is our intention to circumvent this problem and to give more explicit criteria for realizability. Furthermore, EVANS does not give a concrete realization of f_W.

WEISS [1972] was the first one to attack the problem of constructing minimal realizations for certain kinds of i/o maps.

Given a linear i/o map $f_W: B_+^m \to B_+^p$ with unit-pulse response $W(k, l)$, define for a fixed positive integer i the following matrix

$$B_W^{(i)}(k) = \begin{bmatrix} W(k,k-1) & W(k,k-2) & \cdots & W(k,k-i) \\ W(k+1,k-1) & W(k+1,k-2) & \cdots & W(k+1,k-i) \\ \vdots & \vdots & & \vdots \\ W(k+i-1,k-1) & W(k+i-1,,k-2) & \cdots & W(k+i-1,k-i) \end{bmatrix}, \quad k \in Z.$$

(2.2.43)

(2.3.44) THEOREM (WEISS [1972]). Let $f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p$ be a linear input/output map with unit pulse response matrix $W(\cdot, \cdot)$. Assume that f_W has an n-dimensional realization $\Sigma = (F, G, H)$ (see definition (2.3.1)), and for some $N \in Z^+$, $\mathcal{B}_W^{(N)}$ contains a fixed nxn submatrix Γ which is invertible over $\mathcal{B}$. Then there exists a canonical realization over $\mathcal{B}$ of f_W given by

$$\Sigma = (\Gamma_1\Gamma^{-1},\ \Lambda,\ \phi\Gamma^{-1}) \quad ,$$

where

Λ = submatrix of $\mathcal{B}_W^{(N)}$ consisting of those rows of the first block-column whose indices match those of the rows of Γ,

ϕ = submatrix of $\mathcal{B}_W^{(N)}$ consisting of those columns of the first block-row whose indices match those of the columns of Γ,

Γ_1 = the nxn submatrix of $\mathcal{B}_W^{(N+1)}$ whose column positions are the same as the ones of Γ in $\mathcal{B}_W^{(N)}$, and whose i^{th}-row is positioned p-elementary rows below the i^{th} row of Γ.

This algorithm is structurally analogous to the one given by SILVERMAN and MEADOWS [1969] in the realization of time-varying continuous-time systems. The major drawback of the WEISS realization procedure is the unnecessary constraint that there exists an nxn submatrix Γ of $\mathcal{B}_W^{(N)}$ invertible over $\mathcal{B}$. In Chapter Eight, we shall develop an algorithm for construction of canonical realizations which eliminates this constraint.

CHAPTER THREE
LOCAL REALIZABILITY CRITERIA BASED ON THE THEORY OF GENERALIZED REPRESENTATIVE SERIES

In Chapter Two we discussed the importance of the generalized Hankel matrix in finding necessary and sufficient conditions for realizability of either time-invariant or real analytic i/o maps. It was also observed how naturally formal power series arise in the theory of realization for these classes of systems. However, in the discrete time-varying problem, Hankel matrices and power series have either not been fully utilized or have not been considered at all (one exception is the skew power series approach given in the Ph.D. dissertation of POOLLA [1984]).

In this part of the work, we shall first develop some aspects of the theory of generalized representative series. This will be used to derive a realizability criterion based on the concept of the (generalized) Hankel matrix associated with a discrete i/o map. It should be mentioned that these results are the multivariable version of the theory developed by FLIESS [1982, 1983] on realization of "scalar" real analytic i/o maps. Also, it is believed that the introduction of representative formal power series in the theory of discrete-time systems is new.

3.1 Generalized Representative (Formal) Power Series

Let K be any fixed commutative field. Given fixed positive integers m and p, we shall denote by $K^{pxm}[[t,\tau]]$ the K-vector space of all formal power series over K^{pxm} in the two indeterminates t and τ. That is, an element W in $K^{pxm}[[t, \tau]]$ is of the form

$$W = \sum_{i,j>1} W_{ij} t^{i-1} \tau^{j-1} := \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} W_{ij} t^{i-1} \tau^{j-1} \quad , \quad W_{ij} \in K^{pxm} \quad .$$

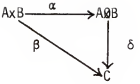
Recall that given K-vector spaces A, B, C, a mapping $f : AXB \to C$ is said to be bilinear if it has the following properties

$$\text{(i)} \qquad f(r_1 a_1 + r_2 a_2, b_1) = r_1 f(a_1, b_1) + r_2 f(a_2, b_1) \quad , \tag{3.1.1}$$

$$\text{(ii)} \qquad f(a_1, r_1 b_1 + r_2 b_2) = r_1 f(a_1, b_1) + r_2 f(a_1, b_2) \quad ,$$

for all a_1, $a_2 \in A$, all b_1, $b_2 \in B$, and all r_1, $r_2 \in K$.

(3.1.2) DEFINITIONS: Let A,B be two K-vector spaces. The tensor product of A and B is a pair $(A \otimes B, \alpha)$ where $A \otimes B$ is a K-vector space, and α is a bilinear mapping of $A \times B \to A \otimes B$ such that for each choice of a K-vector space C, and of a bilinear mapping $\beta : A \times B \to C$, there exists a unique K-homomorphism $\delta : A \otimes B \to C$ for which the diagram



commutes; i.e., $\delta\alpha = \beta$.

Given an integer σ, let us define the following proper subsets of Z:

(3.1.3) $$Z_\sigma^+ = \{k \in Z : k \geq \sigma\} \quad ,$$

(3.1.4) $$Z_\sigma^- = \{k \in Z : k < \sigma - 1\} \quad .$$

Also, for any fixed positive integer m, K^m (K_m) shall denote the K-vector space of all m-element column (row) vectors whose entries are over K. Furthermore, for any K-vector space S and any subset $\bar{Z}$ of Z, $S^{\bar{Z}}$ will represent the K-linear space of all S-valued functions on $\bar{Z}$.

In the sequel λ will denote an arbitrary fixed integer and I will denote a finite set.

Define the following canonical morphism of K-vector spaces

(3.1.5) $$\pi : (K^p)^{Z_\lambda^+} \otimes (K_m)^{Z_\lambda^-} \to (K^{p\times m})^{Z_\lambda^+ \times Z_\lambda^-}$$

and

$$\pi(\sum_{i\in I} f_i \otimes g_i)(k, l) := \sum_{i\in I} f_i(k)g_i(l) \quad .$$

The next result is a generalization of proposition (1.2), Chapter One in HOCHSCHILD [1981].

(3.1.6) PROPOSITION: The canonical morphism $\pi : (K^p)^{Z_\lambda^+} \otimes (K_m)^{Z_\lambda^-} \to (K^{p\times m})^{Z_\lambda^+ \times Z_\lambda^-}$ is injective and its image, $\mathrm{Im}\pi$, consists of all functions h with the property that the K-linear space spanned by the partial functions h_l, where l ranges over Z_λ^- and $h_l(k) := h(k,l)$, is finite-dimensional.

PROOF: Let $\xi = \sum_{j=1}^{r} f_j \otimes g_j$ be any element of the kernel of π, and let V be the K-space of $(K^p)^{Z_\lambda^+}$ generated by $f_1, f_2, \ldots, f_r$. That is,

$$V := \langle f_1, f_2, \ldots, f_r \rangle_K \quad .$$

If $V = (0)$, then the given element of $\ker\pi$ is in the "zero" element of $(K^p)^{Z_\lambda^+} \otimes (K_m)^{Z_\lambda^-}$, and there is nothing to be shown. Otherwise, we choose a basis $\{v_1, v_2, \ldots, v_n\}$ for V. Hence, we have that for each j from $\{1, 2, \ldots, r\}$ there exist scalars $a_{1j}, a_{2j}, \ldots, a_{nj}$ in K such that

$$f_j = \sum_{i=1}^{n} a_{ij} v_i \, .$$

Then

$$\xi = \sum_{j=1}^{r} f_j \otimes g_j = \sum_{j=1}^{r} \left(\sum_{i=1}^{n} a_{ij} v_i \right) \otimes g_j \, ,$$

and by using well-known properties of the tensor product, we obtain

$$\xi = \sum_{i=1}^{n} v_i \otimes h_i \quad , \tag{3.1.7}$$

where

$$h_i := \sum_{j=1}^{r} a_{ij} g_j \quad .$$

Next, we apply π to ξ in (3.1.7) to get

$$\pi(\xi) = \pi\left(\sum_{i=1}^{n} v_i \otimes h_i\right) = \sum_{i=1}^{n} v_i h_i$$

But, note that $\pi(\xi) = 0_{pxm}$ by assumption. Therefore

$$\sum_{i=1}^{n} v_i h_i = [v_1, v_2, \ldots, v_n] \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = 0_{pxm} \quad .$$

Define

$$B := [v_1, v_2, \ldots, v_n]$$

$$H := [h_1^T, h_2^T, \ldots, h_n^T]^T$$

to obtain

$$\pi(\xi) = BH = 0_{\rho xm} \quad .$$

Let l be an arbitrary, but fixed, element of Z_λ^-. Then

$$BH(l) = 0_{\rho xm} \quad .$$

Since the columns of B are linearly independent, we must conclude that

$$H(l) = 0_{nxm} \quad .$$

That is, at each $l \in Z_\lambda^-$

$$h_i(l) = 0 \qquad \text{for } i = 1, 2, \ldots, n \quad .$$

Hence, we have that

$$h_i = 0 \qquad \text{for } i = 1, 2, \ldots, n \quad .$$

By using (3.1.7), we observe then that ξ can be written as

$$\xi = \sum_{i=1}^{n} v_i \otimes 0 = \left(\sum_{i=1}^{n} v_i\right) \otimes 0 \quad .$$

Finally, it is not hard to show by using (3.1.1) that any element in $(K^p)^{Z_\lambda^+} \otimes (K_m)^{Z_\lambda^-}$ of the form $a \otimes 0$ is zero. As a result, we have shown that $\xi = 0$ and π is a monomorphism.

It must be clear from the very definition of π that any h in $\mathrm{Im}\pi$ has the property claimed in the proposition. To complete the proof let us assume that h in $(K^{p\times m})^{Z_\lambda^+ \times Z_\lambda^-}$ has the property. Next we write h as

$$(3.1.8) \qquad h = (h_1, h_2, \ldots, h_m) \quad , \quad \text{where } h_i \in (K^p)^{Z_\lambda^+ \times Z_\lambda^-} \quad .$$

For a fixed element l of Z_λ^- the partial function h_l is given by

$$(3.1.9) \qquad h_l = (h_1(\bullet, l), h_2(\bullet, l), \ldots, h_m(\bullet, l)),$$

$$\text{where } h_i(\bullet, l) \in (K^p)^{Z_\lambda^+}.$$

Let us denote by H the K-space generated by the h_l's; that is

$$H := \langle h_{\lambda-1}, h_{\lambda-2}, h_{\lambda-3}, \ldots \rangle_K \quad .$$

Since H is finite dimensional by assumption so is the K-space spanned by the elementary columns of h_l; that is

$$H_c := \langle h_1(\bullet, \lambda-1), \ldots, h_m(\bullet, \lambda-1) \ , \ h_1(\bullet, \lambda-2),$$

$$\ldots, h_m(\bullet, \lambda-2), \ldots \rangle_K$$

is finite dimensional. Say $\dim H_c = n$, then there exists a basis $\{f_1, f_2, \ldots, f_n\}$ for H_c, where $f_i \in (K^p)^{Z_\lambda^+}$. Therefore, for each $l \in Z_\lambda^-$ and each i from $\{1, 2, \ldots, m\}$, there exist scalars $g_{1i}(l), g_{2i}(l), \ldots, g_{ni}(l)$ such that

$$h_i(\cdot, l) = \sum_{j=1}^{n} f_j \, g_{ji}(l) \quad .$$

By using (3.1.9), we have

$$h_l = \left[\sum_{j=1}^{n} f_j \, g_{ji}(l) \; ; \; \sum_{j=1}^{n} f_j \, g_{j2}(l), \ldots, \sum_{j=1}^{n} f_j \, g_{jm}(l) \right]$$

which can be expressed as

$$h_l = [f_1, f_2, \ldots, f_n] \begin{bmatrix} g_{11}(l) & g_{12}(l) & \cdots & g_{1m}(l) \\ g_{21}(l) & g_{22}(l) & \cdots & g_{2m}(l) \\ \vdots & \vdots & & \vdots \\ g_{n1}(l) & g_{n2}(l) & & g_{nm}(l) \end{bmatrix} \quad .$$

(3.2.10)

Finally, define

$$g_i(l) := [g_{i1}(l); g_{i2}(l); \ldots g_{im}(l)] \quad \text{for } i = 1, 2, \ldots, n \quad .$$

Then (3.1.10) can be rewritten as

$$h_1 = [f_1, f_2, \cdots, f_2] \begin{bmatrix} g_1(1) \\ g_2(1) \\ \vdots \\ g_n(1) \end{bmatrix} = \sum_{i=1}^{n} f_i g_i(1) \quad . \tag{3.1.11}$$

Hence, by choosing coefficients $g_{ji}(1)$ for each $1 \in Z_\lambda^-$, we obtain elements $g_1, g_2, \cdots, g_n$ of $(K_m)^{K_\lambda^-}$ such that (3.1.11) holds. This means

$$h = \pi\left(\sum_{i=1}^{n} f_i \otimes g_i\right) \quad .$$

□

We should note that injectivity of π was pointed out by Fliess [1982], although a proof was not given.

(3.1.12) PROPOSITION. Let m and p be fixed positive integers, and Z_λ^+, Z_λ^- be the proper subsets of Z defined in (3.1.2-3). Then as K-vector spaces

(i) $(K^p)^{Z_\lambda^+}$ is isomorphic to $K^p[[t]]$,

(ii) $(K_m)^{Z_\lambda^-}$ is isomorphic to $K_m[[\tau]]$,

(iii) $(K^{p\times m})^{Z_\lambda^+ \times Z_\lambda^-}$ is isomorphic to $K^{p\times m}[[t, \tau]]$

(iv) $K^p[[t]] \otimes K_m[[\tau]]$ is canonically isomorphic to the sub K-space of $K^{pxm}[[t, \tau]]$ consisting of the following power series

$$(3.1.13) \qquad \sum_{i=1}^{n} a_i(t)b_i(\tau) \quad , \quad a_i(t) \in K^p[[t]], \; b_i(\tau) \in K_m[[\tau]]$$

and n is a finite positive integer.

PROOF: (i) - (iii) are rather obvious while (iv) follows directly from proposition (3.1.6) and from (i) - (iii). □

Any element of $K^p[[t]] \otimes K_m[[\tau]]$, i.e., of the form (3.1.13), will be referred to as a (generalized) representative series (this terminology is due to FLIESS [1982,1983]).

(3.1.14) DEFINITION: Given any formal power series $W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij} t^{i-1}\tau^{j-1}$ in $K^{pxm}[[t, \tau]]$. The characteristic (or representative) matrix C(W) associated with W is the infinite matrix given by

$$(3.1.15) \qquad C(W) = \begin{bmatrix} W_{11} & W_{12} & W_{13} & \cdots \\ W_{21} & W_{22} & W_{23} & \cdots \\ W_{31} & W_{32} & W_{33} & \cdots \\ \vdots & \vdots & \vdots & \\ & & & \end{bmatrix} .$$

Furthermore, the rank of C(W) is defined to be

(i) finite, equal to n, if there is a non-zero minor M of C(W) of order n, and any other minor of C(W) of order (n+1) is zero;

(ii) infinite, otherwise.

Let $R_i(C_i)$ denote the i^{th}-block row (column) of the characteristic matrix C(W); that is, for each $i \in Z^+$.

$$R_i := [W_{i1}, W_{i2}, \cdots]$$

$(C_i := [W_{1i}^T, W_{2i}^T, \ldots.]^T)$, T denotes the transpose operation.

Let R(C) be the set of all finite linear combinations $\sum_i r_i R_i$ $(\sum_i r_i C_i)$, with $r_i \in K$ and $r_i R_i = [r_i W_{i1}, r_i W_{i2}, \ldots]$ $(r_i C_i = [r_i W_{1i}^T, r_i W_{2i}^T, \ldots]^T)$. With componentwise addition and scalar multiplication defined by $rR_i = [rW_{i1}, rW_{i2}, \ldots]$ $(rC_i = [rW_{1i}^T, rW_{2i}^T, \ldots])$, it is very easy to see that R(C) becomes a K-vector space. We shall say that R is the K-vector space spanned by the (infinite) block rows of C(W) while C is the K-vector space spanned by the block columns of C(W).

For each i,j in Z^+ we observe that the i^{th}-block row $R_i = [W_{i1}, W_{i2}, \ldots]$ and the j^{th}-block column $C_j = [W_{ij}^T, W_{2j}^T, \ldots]^T$ of C(W) can be seen respectively as the following formal power series,

(3.1.16) $$W_\tau(i) := \sum_{j=1}^{\infty} W_{ij}\tau^{j-1} \in K^{pxm}[[\tau]] \quad ,$$

(3.1.17) $$W_t(j) := \sum_{i=1}^{\infty} W_{ij}\, t^{i-1} \in K^{pxm}[[t]] \quad .$$

Furthermore, by letting $Q(\tau)(Q(t))$ denote the subspace of $K^{pxm}[[\tau]]$ ($K^{pxm}[[t]]$) consisting of all finite linear combination of the form $\sum_i r_i W_\tau(i)$ ($\sum_j r_j W_t(j)$) with $r_i \in K$ and $r_i W_\tau(i) = \sum_{j=1}^{\infty} (r_i W_{ij})\tau^{j-1}$ ($r_j W_t(j) = \sum_{i=1}^{\infty} (r_j W_{ij})t^{i-1}$), it is not hard to see that $Q(\tau)(Q(t))$ becomes a K-vector space with respect to componentwise addition and scalar multiplication as the ones defined in $K^{pxm}[[\tau]]$ ($K^{pxm}[[t]]$).

It is a small matter now to show that as K-vector spaces "R is isomorphic to $Q(\tau)$" while "C is isomorphic to $Q(t)$".

(3.1.18) PROPOSITION: Given a formal power series $W = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} W_{ij}t^{i-1}\tau^{j-1}$ in $K^{pxm}[[t, \tau]]$, the following statements are equivalent.

(i) W is representative,

(ii) R as a K-vector space is finite-dimensional,

(iii) C as a K-vector space is finite-dimensional.

PROOF: The scalar case (p=m=1) was shown in FLIESS [1982], and we shall prove the general case using a different line of arguments than that given by FLIESS [1982].

First we show that (i) is equivalent to (iii). Recall that we have shown that C_0 is isomorphic to $Q(t)$. By proposition (3.1.12) together with proposition (3.1.6) we have that the K-linear mapping

$$\pi : K^{p}_{[[t]]} \otimes K_m[[\tau]] \to K^{p\times m}_{[[t,\tau]]} : \sum_{i=1}^{n} a_i \otimes b_i \to \sum_{i=1}^{n} a_i b_i$$

is injective, and we have called any element in Imπ a representative series. Furthermore, a $W \in K^{p\times m}[[t, \tau]]$ is in Imπ if and only if the K-linear subspace of $K^{p\times m}[[t]]$ generated by the power series $W_t(j) = \sum_{i=1}^{\infty} W_{ij} t^{i-1}$ with j ranging over Z^+ is finite-dimensional (this is not more than a rephrasing of the last part of proposition (3.1.6)). Next, we use the definition of Q(t) to conclude that $W \in$ Imπ, i.e., it is representative, if and only if Q(t) is finite dimensional. And since $C \simeq Q(t)$, it follows that (i) $\leftrightarrow$ (iii).

On the other hand, we have that a given formal power series

$$W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij} t^{i-1} \tau^{j-1}$$

in $K^{p\times m}[[t, \tau]]$ is representative if, and only if,

$$W^T := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij}^T t^{i-1} \tau^{j-1}$$

in $K^{m\times p}[[t, \tau]]$ is representative (the proof of this is rather obvious so we omit it). And it is not hard to see that $\mathcal{C}(W^T) = \mathcal{C}^T(W)$; therefore, the K-vector space spanned by the block columns of $C(W^T)$, C_0^T, coincides with the K-vector space spanned by the block rows of $\mathcal{C}(W)$, R. Then by using the first part of this proposition, we conclude that (i) and (ii) are equivalent.

□

Now, we are ready to prove the next result.

(3.1.19) THEOREM: A formal power series $W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij} t^{i-1} \tau^{j-1}$ in $K^{pxm}[[t, \tau]]$ is representative if and only if the rank of its characteristic matrix $C(W)$ is finite.

PROOF: Let $X_W^r (X_W^c)$ be the K-linear space spanned by the elementary rows (columns) of $C(W)$. Then it is a well-known fact that the following statements are equivalent.

(i) rank $C(W) = n < \infty$

(ii) X_W^r is finite-dimensional

(iii) X_W^c is finite-dimensional.

But recall that the K-vector space R is finite-dimensional if and only if X_W^r is finite-dimensional. Therefore, by proposition (3.1.18) and (i) - (iii) above, it follows that W is representative if and only if rank $C(W) < \infty$. □

(3.1.20) COROLLARY: Let $W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij} t^{i-1} \tau^{j-1}$ in $K^{pxn}[[t, \tau]]$ be a representative series, i.e., rank $C(W) = n < \infty$. Then, there is a decomposition of W of length n, that is,

$$W = \sum_{v=1}^{n} a_v(t) b_v(\tau) \quad ,$$

where $a_v(t) \in K^p[[t]]$, and $b_v(\tau) \in K_m[[\tau]]$. Furthermore, for any other decomposition of W, say

$$W = \sum_{\nu=1}^{n'} a'_\nu(t) b'_\nu(\tau) \quad ,$$

with $a'_\nu(t) \in K^p[[t]]$, $b'_\nu(\tau) \in K_m[[\tau]]$, we have $n' \geq n$.

PROOF: Assume that rank $C(W) = n < \infty$. Then, it is a well-known fact that X_W^r, the K-vector space generated by the elementary rows of $C(W)$, has dimension n. That means that there exists a basis $r_{k(1)}, r_{k(2)}, \ldots, r_{k(n)}$ for X_W^r, where r_k denotes the k^{th}-elementary row of $C(W)$. Hence, for each $j \in Z^+$ there exists scalars $\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}$ in K, such that the j^{th}-elementary row of $C(W)$ can be expressed uniquely as

$$(3.1.21) \qquad r_j = \sum_{\mu=1}^{n} \alpha_\mu^{(j)} r_{k(\mu)} \quad .$$

Next, write the i^{th}-block row of $C(W)$ as

$$R_i = \left[\begin{array}{cccc|cccc} \omega_{11}^{(i1)} & \omega_{12}^{(i1)} & \cdots & \omega_{1m}^{(i1)} & \omega_{11}^{(i2)} & \omega_{22}^{(i2)} & \cdots & \omega_{1m}^{(i2)} & \cdots \\ \omega_{21}^{(i1)} & \omega_{22}^{(i1)} & \cdots & \omega_{2m}^{(i1)} & \omega_{21}^{(i2)} & \omega_{21}^{(i2)} & \cdots & \omega_{2m}^{(i2)} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \\ \omega_{p1}^{(i1)} & \omega_{p2}^{(i1)} & \cdots & \omega_{pm}^{(i1)} & \omega_{p1}^{(i2)} & \omega_{p2}^{(i2)} & \cdots & \omega_{pm}^{(i2)} & \cdots \end{array}\right] \quad .$$

(3.1.22)

By using (3.1.21), we have that the ν^{th}-elementary row $r_{i\nu}$ of R_i can be

written as

$$(3.1.23) \qquad r_{iv} = \sum_{\mu=1}^{n} \alpha_{\mu}^{(iv)} r_{k(\mu)} \qquad v = 1, 2, \ldots, p \quad .$$

Next, for each $\mu \in \{1, 2, \ldots, n\}$ write $r_{k(\mu)}$ as

$$(3.1.24) \qquad r_{k(\mu)} = [q_{k(\mu)}^{1}, q_{k(\mu)}^{2}, q_{k(\mu)}^{3}, \ldots] \text{ with } q_{k(\mu)}^{i} \in K_m \quad .$$

Hence, by putting (3.1.24) into (3.1.23) we get

$$r_{iv} = [\sum_{\mu=1}^{n} \alpha_{\mu}^{(iv)} q_{k(\mu)}^{i}, \sum_{\mu=1}^{n} \alpha_{\mu}^{(iv)} q_{k(\mu)}^{2}, \ldots] \quad \text{for } i \in Z^{+},$$

$$v \in \{1, 2, \ldots, p\} \quad .$$

And by substituting (3.1.25) in (3.1.22), it is easy to see that

$$W_{ij} = \begin{bmatrix} \sum_{\mu=1}^{n} \alpha_{\mu}^{(i1)} q_{k(\mu)}^{j} \\ \sum_{\mu=1}^{n} \alpha_{\mu}^{(i2)} q_{k(\mu)}^{j} \\ \vdots \\ \sum_{\mu=1}^{n} \alpha_{\mu}^{(ip)} q_{k(\mu)}^{j} \end{bmatrix} \qquad \text{for each } i,j \in Z^{+} \quad .$$

Define for each $i \in Z^{+}$, and each $\mu \in \{1, 2, \ldots, n\}$ the vector $a_{\mu}^{(i)}$

in K^p by

$$a_\mu^{(i)} := \begin{bmatrix} \alpha_\mu^{(i1)} \\ \alpha_\mu^{(i2)} \\ \vdots \\ \alpha_\mu^{(ip)} \end{bmatrix} \quad \text{(recall that } \alpha_\mu^{(i1)} \in K\text{)} \quad .$$

Then we can write W_{ij} as

$$W_{ij} = \sum_{\mu=1}^{n} a_\mu^{(i)} q_{k(\mu)}^j \quad .$$

As a result, the power series $W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij} t^{i-1} \tau^{j-1}$ can be expressed as

$$W = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{\mu=1}^{n} a_\mu^{(i)} q^j{}_{k(\mu)} \right) t^{i-1} \tau^{j-1}$$

which can be written, after an interchange of summations, as

$$W = \sum_{\mu=1}^{n} \left(\sum_{i=1}^{\infty} a_\mu^{(i)} t^{i-1} \right) \left(\sum_{j=1}^{\infty} q_{k(\mu)}^j \tau^{j-1} \right)$$

$$= \sum_{\mu=1}^{n} a_\mu(t) b_\mu(\tau)$$

where

$$a_\mu(t) := \sum_{i=1}^{\infty} a_u^{(i)} t^{i-1} \in K^p[[t]]$$

$$b_\mu(\tau) := \sum_{j=1}^{\infty} q_{k(\mu)}^j \in K_m[[\tau]] \quad \text{(see 3.1.24))} \quad .$$

We have thus shown that in fact W can be decomposed into n terms. The last part of the corollary follows from the fact that a decomposition of W with $n' < n$ would imply that rank $C(W) = n' < n$, which clearly is a contradiction. (To see this, just reverse the steps given in the proof above.)

□

3.2 Local Realizability Criterion

For a given integer λ, we defined by CP_λ the proper subset of CP given by

$$CP_\lambda = \{(k, l \in Z \times Z : k > \lambda > l\} \quad .$$

Denote by $[K^{p\times m}]^{CP_\lambda}$ the K-vector space consisting of all functions with domain CP_λ, and with values in the K-vector space $K^{p\times m}$.

Consider a linear input/output map $f_W : B_+^m \to B_+^p$ with unit pulse response W(k, l), (k, l) $\in$ CP. And for each fixed integer λ define the following mapping

$$(3.2.1) \qquad \Phi_\lambda : [K^{p\times m}]^{CP_\lambda} \to K^{p\times m}[[t, \tau]] : W \to \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij}(\lambda) t^{i-1} \tau^{j-1}$$

where

$$(3.2.2) \qquad W_{ij}(\lambda) := W(\lambda+i-1, \lambda-j) \quad \text{for all } i,j \in Z^+ \quad .$$

(3.2.3) PROPOSITION: The mapping $\Phi_\lambda : [K^{pxm}]^{CP_\lambda} \to K^{pxm}[[t, \tau]]$ defined by (3.2.1) is a K-linear isomorphism. Therefore, as K-vector spaces

$$[K^{pxm}]^{CP_\lambda} \cong K^{pxm}[[t, \tau]] \quad .$$

PROOF: That Φ_λ is a linear and one-to-one can be seen directly from the definition of addition and scalar multiplication in the K-vector spaces $[K^{pxm}]^{CP_\lambda}$ and $K^{pxm}[[t, \tau]]$. Let us show that Φ_λ is onto. For any element

$$\sum_{i,j\geq 1} \Omega_{ij}(\lambda) t^{i-1} \tau^{j-1}$$

in $K^{pxm}[[t, \tau]]$ define

$$W(k, l) := \Omega_{k-\lambda, -l+\lambda-1} \quad \text{for all } (k, l) \in CP_\lambda.$$

Clearly $W \in [K^{pxm}]^{CP_\lambda}$, and it is such that

$$\Phi_\lambda(W) = \sum_{i,j} \Omega_{ij}(\lambda) \lambda^i \tau^j .$$

□

The above result although simple in nature will bring important consequences in our work, since it will allow us to study factorizability of unit-pulse response matrices W over CP_λ by using the theory of representative power series.

(3.2.4) PROPOSITION: Let $W \in [K^{pxm}]^{CP_\lambda}$ be such that $W(k, l) = f(k)g(l)$ for all (k, l) in CP_λ, where $f : \{\lambda, \lambda+1, \ldots\} \to K^p$ and $g : \{\ldots, \lambda-2, \lambda-1\} \to K_m$. Then $\phi_\lambda(W)$ is a representative formal power series.

PROOF: By definition $\phi_\lambda(W) = \sum_{i,j\geq 1} W(\lambda+i-1, \lambda-j)t^{i-1}\tau^{j-1}$. By hypothesis, we have that

$$W(\lambda+i-1, \lambda-j) = f(\lambda+i-1)g(\lambda-j) \quad \text{for} \quad i, j \in Z^+ \quad .$$

Hence

$$\phi_\lambda(W) = \sum_{i,j\geq 1} f(\lambda+i-1)g(\lambda-j)t^{i-1}\tau^{j-1}$$

$$= \left(\sum_{i=1}^{\infty} f(\lambda+i-1)t^{i-1}\right)\left(\sum_{j=1}^{\infty} g(\lambda-j)\tau^{j-1}\right) \quad .$$

That is

$$\phi_\lambda(W) = a_\lambda(t)b_\lambda(t)$$

where

$$a_\lambda(t) = \sum_{i=1}^{\infty} f(\lambda+1-1)t^{i-1} \in K^p[[t]] \ ;$$

$$b_\lambda(\tau) = \sum_{j=1}^{\infty} g(\lambda-j)\tau^{j-1} \in K_m[[\tau \quad .$$

Therefore, $\psi_\lambda(W)$ is representative by definition (see (3.1.13)).

□

Basically, the above proposition says that if $W(\cdot, \cdot)$ is separable over CP_λ, then $\psi_\lambda(W)$ is representative. It is natural to see whether or not the reversed implication is also true.

(3.2.5) PROPOSITION: Let $a(t) = \sum_{i=1}^{\infty} a_i(\lambda)t^{i-1} \in K^p[[t]]$, and $b(\tau) = \sum_{j=1}^{\infty} b_j(\lambda)\tau^{j-1} \in K_m[[\tau]]$, then we have that

(i) $W := a(t)b(\tau) \in K^{pxm}[[t, \tau]]$,

(ii) $\psi_\lambda^{-1}(W)(k, l) = f(k)g(l)$ for all $(k, l) \in CP_\lambda$,

where

$$f : \{\lambda, \lambda+1, \ldots\} \to K^p : k \to a_{k-\lambda+1}(\lambda) \quad ,$$

$$g : \{\ldots, \lambda-2, \lambda-1\} \to K_m : l \to b_{\lambda-l}(\lambda) \quad .$$

PROOF: (i) is clear since $a(t)b(\tau) = \sum_{i,j=1}^{\infty} a_i(\lambda)b_j(\lambda)t^{i-1}\tau^{j-1} \in$

$K^{pxm}[[t, \tau]]$. To show (ii), observe that

$$\phi_\lambda(fg) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(\lambda+i-1)g(\lambda-j)t^{i-1}\tau^{j-1} \quad \text{(see (3.2.1))}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i(\lambda)b_j(\lambda)t^{i-1}\tau^{j-1} \quad \text{(by definition of f and g)}$$

$$= \Big(\sum_{i=1}^{\infty} a_i(\lambda)t^{i-1}\Big)\Big(\sum_{j=1}^{\infty} b_j(\lambda)\tau^{j-1}\Big)$$

$$= a(t)b(\tau) \quad .$$

But recall that ϕ_λ is a vector isomorphism, then

$$[\phi_\lambda^{-1}(a(t)b(\tau))](k, l) = f(k)g(l) \quad \text{for all } (k, l) \in CP_\lambda \quad .$$

□

It is important to realize that ϕ_λ is <u>not</u> a ring homomorphism in the case where p = m. To see that, let p=m=1 and consider the functions f and g defined in proposition (3.2.5). Then

$$\phi_\lambda(f) = \Big(\sum_{i=1}^{\infty} a_i(\lambda)t^{i-1}\Big)\Big(\sum_{j=1}^{\infty} \tau^{j-1}\Big) \; ; \quad \phi_\lambda(g) = \Big(\sum_{i=1}^{\infty} t^{i-1}\Big)\Big(\sum_{j=1}^{\infty} o_j\tau^{j-1}\Big).$$

Clearly,

$$\phi_\lambda(fg) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i(\lambda) b_j(\lambda) t^{i-1} \tau^{j-1} \neq \phi_\lambda(f)\phi_\lambda(g) \quad .$$

(3.2.6) THEOREM: Let $W \in [K^{p \times m}]^{CP_\lambda}$. Then $W(k,l) = \sum_{s=1}^{n} f^{(s)}(k) g^{(s)}(l)$ for all $(k, l) \in CP$ for some $f^{(s)} : \{\lambda, \lambda+1, \ldots\} \to K^p$, $g^{(s)} : \{\ldots, \lambda-2, \lambda-1\} \to K_m$ if, and only if

$$\phi_\lambda(W) = \sum_{s=1}^{n} a_s(t) b_s(\tau)$$

for some $a_s(t) \in K^p[[t]]$ and $b_s(\tau) \in K_m[[\tau]]$.

PROOF: First, let us assume that W can be written as

$$W(k, l) = \sum_{s=1}^{n} f^{(s)}(k) g^{(s)}(l) \quad \text{for all } (k, l) \in CP_\lambda \quad .$$

Then by definition of ϕ_λ

$$\phi_\lambda(W) = \sum_{i,j>1} W(\lambda+i-1, \lambda-j) t^{i-1} \tau^{j-1} = \sum_{i,j>1} \left(\sum_{s=1}^{n} f^{(s)}(\lambda+i-1) g^{(s)}(\lambda-j) \right) t^{i-1} \tau^{j-1}.$$

Next define

$$a_s(t) = \sum_{i=1}^{\infty} f^{(s)}(\lambda+i-1) t^{i-1} \in K^p[[t]];$$

$$b_s(\tau) = \sum_{j=1}^{\infty} g^{(s)}(\lambda-j) \tau^{j-1} \in K_m[[\tau]]$$

to obtain

$$\phi_\lambda(W) = \sum_{s=1}^{n} a_s(t) b_s(\tau) \quad .$$

Conversely, assume that

$$\Phi_\lambda(W) = \sum_{s=1}^{n} a_s(t)b_s(\tau) \quad \text{with} \quad a_s(t) \in K^p[[t]],\ b_s(\tau) \in K_m[[\tau]] \ .$$

Because Φ_λ^{-1} is also a K-linear homomorphism, we get

$$W(k, l) = \Phi_\lambda^{-1}(\Phi_\lambda(W))(k, l) = \Phi_\lambda^{-1}\Big(\sum_{s=1}^{n} a_s(t)b_s(\tau)\Big) =$$

$$= \sum_{s=1}^{n} \Phi_\lambda^{-1}(a_s(t)b_s(\tau))$$

$$= \sum_{s=1}^{n} f^{(s)}(k)g^{(s)}(l) \quad \text{for all } (k, l) \in CP_\lambda . \quad \text{(By prop. (3.2.5))}$$

□

Next, we relate the above results with the realizability of f_W : $B_+^m \to B_+^p$ on CP_λ.

(3.2.7) COROLLARY: An i/o map $f_W: B_+^m \to B_+^p$ with unit-pulse response matrix $W(\cdot, \cdot)$ is realizable on CP_λ if and only if $\Phi_\lambda(W)$ is a representative power series.

PROOF: By proposition (2.3.38) f_W is realizable on CP_λ if, and only if there exist matrix functions

$$P_\lambda(\cdot) : \{\lambda, \lambda+1, \ldots\} \to K^{p\times n}$$

, n a fixed positive integer

$$Q_\lambda(\cdot) : \{\ldots, \lambda-2, \lambda-1\} \to K^{n\times m}$$

such that

$$W(k, l) = P_\lambda(k)Q_\lambda(l) \quad \text{for all } (k, l) \in CP_\lambda.$$

Next, we write P_λ and Q_λ as

$$P_\lambda = [p_\lambda^{(1)} ; p_\lambda^{(2)} ; \ldots; p_\lambda^{(n)}] \quad ,$$

$$Q_\lambda = \begin{bmatrix} q_\lambda^{(1)} \\ q_\lambda^{(2)} \\ \vdots \\ q_\lambda^{(n)} \end{bmatrix} \quad ,$$

where $p_\lambda(s) : \{\lambda, \lambda+1, \ldots\} \to K^p$;

$$q_\lambda^{(s)} : \{\ldots, \lambda-2, \lambda-1\} \to K_m \quad .$$

Therefore

$$(3.2.8) \qquad W(k,l) = \sum_{s=1}^{n} \rho_\lambda^{(s)}(k) q_\lambda^{(s)}(l) \quad \text{for all } (k,\ l) \in CP_\lambda,$$

and by theorem (3.2.6), any $W \in [K^{p\times m}]^{CP_\lambda}$ can be expressed as in (3.2.8) if and only if $\phi_\lambda(W) = \sum_{s=1}^{n} a_s(t) b_s(\tau)$ with $a_s(t) \in K^p[[t]]$, $b_s(\tau) \in K_m[[\tau]]$. That is, if and only if $\phi_\lambda(W)$ is representative.

□

(3.2.9) THEOREM: An input/output map $f_W : B_+^m \to B_+^p$ is realizable on CP_λ if, and only if, the rank of the characteristic matrix associated with $\phi_\lambda(W) = \sum_{i,j\geq 1} W(\lambda+i-1,\ \lambda-j) t^{i-1} \tau^{j-1}$ is finite. That is, if and only if

$$C^\lambda(W) := \begin{bmatrix} W(\lambda,\lambda-1) & W(\lambda,\lambda-2) & W(\lambda,\lambda-3) & \cdots \\ W(\lambda+1,\lambda-1) & W(\lambda+1,\lambda-2) & W(\lambda+1,\lambda-3) & \cdots \\ W(\lambda+2,\lambda-1) & W(\lambda+2,\lambda-2) & W(\lambda+2,\lambda-3) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

has finite rank.

PROOF: From corollary (3.2.7) f_W is realizable on CP_λ if, and only if, $\phi_\lambda(W)$ is a representative series. On the other hand, by theorem (3.1.19) $\phi_\lambda(W)$ is a representative power series if, and only if, $C^\lambda(W)$ has finite rank.

□

So far, we have found a realizability criteria for f_W on each piece CP_λ. Then the most natural thing to do is to link realizability on each

CP_λ to find necessary and sufficient conditions for f_W to be realizable on CP.

(3.2.10) THEOREM: An i/o map $f_W : R_+^m \to R_+^p$ with unit-pulse response matrix $W(\cdot, \cdot)$ is realizable on the causal half-plane CP if, and only if there exists a finite positive integer n such that

(3.2.11) $\quad$ rank $C^\lambda(W) \leq n$ for each $\lambda \in Z$.

In fact, if we let $n_\lambda :=$ rank $C^\lambda(W)$, then $n :=$ maximum $\{n_\lambda : \lambda \in Z\}$ is the dimension of a minimal realization $\Sigma = (D, E, H)$ of f_W on CP, and n is the smallest positive integer for which (3.2.11) is satisfied.

PROOF: (Necessity). Suppose f_W has an n-dimensional realization $\Sigma = (D, E, H)$ on CP. Then by definition

$$W(k, l) = H(k)\phi(k, l+1)E(l+1) \quad , \text{ for all } (k, l) \in CP.$$

For each $\lambda \in Z$, define

$$P_\lambda(k) = H(k)\phi(k, \lambda+1) \quad , \quad k \geq \lambda \quad ,$$

$$Q_\lambda(l) = \phi(\lambda, l+1)E(l+1) \quad , \quad \lambda > l \quad .$$

Then

$$W(k, l) = P_\lambda(k)Q_\lambda(l) \quad , \quad (k, l) \in CP_\lambda \quad .$$

Next, write P_λ and Q_λ as we did in the proof of corollary (3.2.7) to express W as

$$W(k, l) = \sum_{s=1}^{n} p_\lambda^{(s)}(k)q_\lambda^{(s)}(l) \quad , \quad \text{for all } (k, l) \in CP.$$

Hence, $\phi_\lambda(W)$ is representative and by corollary (3.1.20) we have that

$$\text{rank } C^\lambda(W) = n_\lambda \leq n \quad , \quad \text{for each } \lambda \in Z.$$

Conversely, let us assume that for each $\lambda \in Z$, rank $C^\lambda(W) \leq n$, where n is finite positive integer independent of λ. Let λ_1 be a fixed integer, then by hypothesis

$$\text{rank } C^{\lambda_1}(W) = n_1 \leq n \quad .$$

By corollary (3.1.20) there is a decomposition $\phi_{\lambda_1}(W)$ in n_1 elements; that is,

$$\phi_{\lambda_1}(W) = \sum_{s=1}^{n_1} a_s(t)b_s(\tau) \quad , \quad a_s(t) \in K^p[[t]], \; b_s(\tau) \in K_m[[\tau]].$$

By theorem (3.2.6)

$$W(k,l) = \sum_{s=1}^{n_1} f_{\lambda_1}^{(s)}(k) g_{\lambda_1}^{(s)}(l), \quad \text{for all } (k, l) \in CP_{\lambda_1} .$$

Define

$$\hat{P}_{\lambda_1} := (f_{\lambda_1}^{(1)} ; f_{\lambda_1}^{(2)}; \dots ; f_{\lambda_1}^{(n_1)})$$

$$\hat{Q}_{\lambda_1} := \begin{bmatrix} g_{\lambda_1}^{(1)} \\ g_{\lambda_1}^{(2)} \\ \vdots \\ g_{\lambda_1}^{(n_1)} \end{bmatrix} .$$

Introducing $(n-n_1)$ superfluous states we can obtain an n-dimensional realization $\Sigma = (D, E, H)$ of f_W on CP_{λ_1}. In fact, we can take

$$D = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{(n-n_1)} \end{bmatrix} , \quad E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} , \quad H = [H_1, 0] ,$$

where $E_1(k) = Q_{\lambda_1}(k-1)$, for $k \leq \lambda$ and $E_1(k) = 0$ for $k > \lambda$; $H_1(k) = P_{\lambda_1}(k)$ for $k \geq \lambda$ and $H_1(k) = 0$ for $k < \lambda$.

We have thus shown that for each λ_1 in Z f_W is realizable on CP_{λ_1} by an n-dimensional system $\Sigma = (D, E, H)$, and n is independent of λ. And finally by using proposition (2.3.42), we conclude that f_W is realizable on CP. The last statement of the theorem follows from the definition of n, and from the proof above. □

(3.2.12) DEFINITION: Given a linear i/o map $f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p$ with unit pulse response matrix $W(k, l)$, $(k, l) \in CP$, define as its (generalized) Hankel matrix B_W the infinite matrix given by

$$(3.2.13)\quad B_W(k) := \begin{bmatrix} W_{11}(k) & W_{12}(k) & W_{13}(k) & \cdots \\ W_{21}(k) & W_{22}(k) & W_{23}(k) & \cdots \\ W_{31}(k) & W_{32}(k) & W_{33}(k) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \text{ for each } k \in Z$$

where

$$(3.2.14)\quad W_{ij}(k) := W(k+i-1, k-j) \text{ for each } i, j \in Z^+, \text{ and each } k \in Z.$$

Now, let k be a fixed arbitrary integer. It is straightforward to see that the characteristic matrix $C^k(W)$ associated with

$$\Phi_k(W) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W(k+i-1, k-j) t^{i-1} \tau^{j-1}$$

in theorem (3.2.9) coincides with $B_w(k)$. Therefore, theorem (3.2.10) can be stated as follows.

(3.2.15) THEOREM: An input/output map $f_w : B_+^n \to B_+^p$ with unit-pulse response $W(\cdot, \cdot)$ is realizable (on the causal-half plane CP) if and only if there exists a finite positive integer n such that

$$\text{rank } B_w(k) \leq n \quad \text{for all } k \in Z \quad . \tag{3.2.16}$$

Furthermore, the smallest positive integer n for which (3.2.16) holds is the dimension of a minimal realization.

This criteria for realizability can be considered as the discrete time counterpart of theorem (2.2.12) in the sense that it is a one-variable criterion, and that it requires the computation of the rank of a collection of infinite matrices, B_w, indexed by Z. However, there is a big difference because in (3.2.16) equality may not be attained in general as we now show.

(3.2.7) EXAMPLE: Consider $f_w : B_+ \to B_+$ whose unit-pulse response is given by

$$w(k, l) = \begin{cases} 1, \text{ for } k = 5, 6, \text{ and } l = 3, 4, \\ 0, \text{ otherwise.} \end{cases} \tag{3.2.18}$$

Note that

$$w(k, l) = [\Delta(k-5) + \Delta(k-6)][\Delta(l-3) + \Delta(l-4)] \quad \text{for all } (k, l) \in CP.$$

Because $w(\cdot, \cdot)$ is separable, it must be clear that f_w is realizable on CP.

In this case

$$B_w(k) = \begin{bmatrix} w(k, k-1) & w(k, k-2) & \cdots \\ w(k+1, k-1) & w(k+1, k-2) & \cdots \\ w(k+2, k-1) & w(k+2, k-2) & \cdots \\ \vdots & \vdots & \end{bmatrix}, \quad k \in Z.$$

It is very simple to see that

$$B_w(k) \equiv 0 \quad \text{for all } k \in Z - \{4, 5, 6\},$$

while

$$B_w(4) = \begin{bmatrix} 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & \vdots & \end{bmatrix}, \quad B_w(5) = \begin{bmatrix} 1 & 1 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ & & & \\ \vdots & \vdots & \vdots & \end{bmatrix}, \quad B_w(6) = \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ & & & & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Thus,

$$\text{rank } B_w(k) = \begin{cases} 1, & k = 4, 5, 6 \\ 0, & \text{otherwise} \end{cases}$$

and we observe that f_w is realizable (by theorem (3.2.15)); however, there is no positive integer n such that rank $B_w(k) = n$ for each $k \in Z$.

CHAPTER FOUR
GLOBAL REALIZATION CRITERIA

It was seen in Chapter Two that the realizability criteria for time-varying real analytic input/output maps can be viewed as generalizations of criteria for time-invariant response maps. As an example, from theorem (2.1.9) we have that a constant i/o map $f_W = \{W_1, W_2, \ldots\}$ is realizable if and only if the formal power series $W(z^{-1}) = \sum_{i=1}^{\infty} W_i z^{-i}$ in $K^{pxm}[[z^{-1}]]$ is recurrent (see definition (2.1.8)). And it was shown in KAMEN [1979] that a generalization of this condition gives us a realizability criterion for analytic i/o maps $f_W : C^m(J) \to C^p(J)$.

In this part of the work, we shall show that for discrete time-varying response maps, it is possible to accomplish also the above mentioned generalization. In point of fact our results are similar to the ones obtained by KAMEN [1979] for the analytic case. However, it is worthwhile to point out that due to the differences in structure of the rings A(J) and B, e.g., the former is a Bezout domain while the latter is not even an integral domain, there is no reason why we should expect that those criteria for realizability in the analytic framework carry over the discrete-time one. On the other hand, discrete time-varying systems and analytic systems have indeed something in common, and that

is that both have an associated pseudo-linear transformation (see equations (2.2.23) and (2.3.28)). This fact is a good reason for the existing similarity between the two mentioned frameworks.

Consider a linear i/o map $f_W : B_+^m \to B_+^p : u \to f_W(u) : k \to \sum_{l=-\infty}^{k} W(k, l)u(l)$ (recall that W is the unit-pulse response matrix of f_W). For each fixed positive integers i,j, associate with f_W the following matrix.

$$W_{ij}(k) := W(k+i-1, k-j) \quad , \quad k \in Z. \tag{4.1}$$

Clearly, each W_{ij} is a pxm matrix over the ring B. It is very easy to show that f_W is realizable (on CP = causal half-plane) by a time-invariant system $\Sigma = (D, E, H)$ (that is, D, E, H are constant matrices) if, and only if

(i) the matrices W_{ij} are constant over Z,

(ii) $W_{1,2}(k) = W_{i,1}(k) = HD^{i-1}E$ for all $i \in Z^+$, and all $k \in Z$. (4.2)

We next give necessary and sufficient conditions for realizability of f_W which reduce to (4.2) when there is a time-invariant realization.

(4.3) LEMMA: Given an n-dimensional system $\Sigma = (D, E, H)$ over B, let $W_\Sigma(\cdot, \cdot)$ denote its pulse-response matrix, that is, $W_\Sigma(k, l) =$

$H(k)\phi_\Sigma(k, l+1)E(l+1)$ for all $(k, l) \in CP$. Then for each fixed positive integers i,j

$$(W_\Sigma)_{ij} = (\hat{S}^{i-1}H^T)^T(S^{j-1}E) \quad ,$$

where

(i) $(W_\Sigma)_{ij}(k) := W_\Sigma(k+i-1, k-j) \quad , \quad k \in Z$

(ii) $\hat{S}^{i-1}$ is the (i-1)-fold composition of $\hat{S}$ as defined by (2.3.31), and S^{j-1} is the (j-1)-fold composition of S as given by (2.3.28).

PROOF: By using (2.3.31), it can be shown by induction on i that

$$(\hat{S}^{i-1}H^T)(k) = H(k+i-1)D(k+i-1)D(k+i-2) \ \ldots \ D(k+2)D(k+1) \tag{4.4}$$

for each $i \in Z^+$, and each $k \in Z$.

From the definition of the state-transition matrix $\phi_\Sigma(\cdot, \cdot)$ of D (see (2.3.5)), we observe that (4.4) can be written as

$$(\hat{S}^{i-1}H^T)^T(k) = H(k+i-1)\phi_\Sigma(k+i-1, k) \quad , \quad i \in Z^+ \ , \quad k \in Z. \tag{4.5}$$

In similar manner, by using (2.3.28), we can show that

$$(S^{j-1}E)(k) = \phi_\Sigma(k, k-j+1)E(k-j+1) \tag{4.6}$$

for each $j \in Z^+$, and each $k \in Z$.

Next combine (4.5) and (4.6) to get

$$(\hat{S}^{i-1}H^T)^T(k)(S^{j-1}E)(k) = H(k+i-1,k)\phi_\Sigma(k,k-j+1)E(k-j+1) \tag{4.7}$$

for $i, j, \in Z^+$, $k \in Z$.

Finally, use the semigroup property of ϕ_Σ (that is $\phi(k, l_1)\phi(l_1, l) = \phi(k, l)$ for $k \geq l_1 \geq l$) and the definition of W_Σ to obtain

$$(\hat{S}^{i-1}H^T)^T(S^{j-1}E) = (W_\Sigma)_{ij} \quad , \quad i, j \in Z^+.$$

□

Using lemma (4.3) we get the following necessary and sufficient conditions for realizability in terms of the W_{ij} given by (4.1).

(4.8) PROPOSITION: Given a response map $f_W: B_+^m \to B_+^p$ with unit-pulse response matrix $W(\cdot, \cdot)$ and an n-dimensional system $\Sigma = (D, E, H)$ over B, the following statements are equivalent:

(i) $\Sigma = (D, E, H)$ realizes f_W,

(ii) $W_{ij} = (\hat{S}^{i-1}H^T)^T(S^{j-1}E)$, for each $i,j \in Z^+$,

(iii) $W_{1j} = H(S^{j-1}E)$, for each $j \in Z^+$,

(iv) $W_{i1} = (\hat{S}^{i-1}H^T)^T E$, for each $i \in Z^+$.

PROOF: That (i) implies (ii) follows directly from the definition of realization ($W = W_\Sigma$) and from lemma (4.3). Also, that (ii) implies (iii) and (iv) is rather clear since $\hat{S}^o = S^o =$ identity operator. So, we start by showing that (ii) implies (i).

Note that at each $(k, l) \in CP$, $W(\cdot, \cdot)$ can be expressed as

$$W(k, l) = \sum_{j=1}^{\infty} W_{1j}(k)\Delta(k-l-j) \tag{4.9}$$

where

$$\Delta(k) = \begin{cases} 1 & , \quad k = 0 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad \text{(The unit-pulse sequence).}$$

By assumption $W_{1j} = HS^{j-1}E$, then (4.9) becomes

$$W(k, l) = \sum_{j=1}^{\infty} H(k)(S^{j-1}E)(k)\Delta(k-l-j) \quad \text{for each } (k, l) \in CP. \tag{4.10}$$

By the same argument and by using lemma (4.3), it is easy to show that at each $(k, l) \in CP$, $W_\Sigma(k, l)$ (the unit-pulse response of Σ) can be expressed as

$$W_\Sigma(k, l) = \sum_{j=1}^{\infty} H(k)(S^{j-1}E)(k)\Delta(k-l-j) \quad . \tag{4.11}$$

By comparing (4.10) and (4.11) we conclude that $W(k, l) = W_\Sigma(k, l)$, for each $(k, l) \in CP$; that is Σ realizes f_W.

Finally, by noticing that $W(\cdot, \cdot)$ can also be written as

$$W(k, l) = \sum_{i=1}^{\infty} W_{il}(l+1)\Delta(k-l-i) \quad \text{at each} \quad (k, l) \in CP$$

and by the assumption $W_{il} = (\hat{S}^{i-1}H^T)^T E$, $i \in Z^+$, we can show (iv) implies (i).

□

Here, it is important to emphasize that the above result reduces into (4.2) when the system $\Sigma = (D, E, H)$ is constant.

It is a well-known fact that the i/o map $f_W : B_+^m \to B_+^p$ has a time-invariant realization if, and only if

(i) the W_{ij}'s are time-independent for each $i,j \in Z^+$,

(ii) the W_{ij}'s or W_{ji}'s satisfy a recurrence relation over the field K; that is, there exist scalars $b_0, b_1, \cdots, b_{q-1}$ in K such that

$$(4.12) \qquad W_{q+1,j} = \sum_{i=0}^{q-1} b_i W_{i+1,j} \quad \text{for all} \quad j \in Z^+ \quad .$$

We shall show next that in the discrete time varying case, there are conditions for realizability which resemble (4.12). In fact, it is interesting to observe that those results together with proposition (4.8) are the discrete-time versions of the ones obtained by KAMEN [1979] in the continuous-time analytic case.

In the sequel, we shall need the following basic fact.

(4.13) LEMMA: $f_W : B_+^m \to B_+^p$ with unit-pulse response $W(\cdot, \cdot)$. Then, for each fixed $i,j \in Z^+$

$$\sigma W_{i+1,j} = W_{i,j+1}$$

where σ = right-shift operator, and W_{ij} is defined by (4.1).

PROOF: Just use the corresponding definitions to justify each step in

$$(\sigma W_{i+1,j})(k) = W_{i-1,j}(k-1) = W(k-1+i-1, k-1-j) =$$

$$W(k+1-1,\ k-(j+1)) = W_{i,j+1}(k) \quad , \quad k \in Z \quad .$$

□

(4.14) THEOREM: An i/o map $f_W : B_+^m \to B_+^p$ with unit-pulse response $W(\cdot, \cdot)$ is realizable by an observable system $\Sigma = (D, E, H)$ over B (that is, rank $[H^T, \hat{S}H^T, \ldots \hat{S}^{N-1}H^T](k) = \dim \Sigma$ at each $k \in Z$, for some $N \in Z^+$) if, and only if there exist pxp matrices $P_0, P_1, \ldots, P_{q-1}$ over B such that

$$W_{q+1,\,j} = \sum_{i=1}^{q-1} P_i W_{i+1,j}, \quad \text{for all} \quad j \in Z^+. \tag{4.15}$$

PROOF: (Necessity). Suppose that the i/o map $f_W : B_+^m \to B_+^p$ has an observable realization $\Sigma = (D, E, H)$ over B. Then by theorem (2.3.33)

$$\text{rank } O_N^T(k) = \text{rank}[H^T, \hat{S}H^T, \ldots \hat{S}^{N-1}H^T](k) = n, \quad \text{for all } k \in Z,$$

where N is some positive integer.

That is equivalent to say that the nxn matrix $O_N^T O_N$ is invertible over B. Let $\bar{O}$ be the px Np matrix over B defined by

$$\bar{O} = [\hat{S}^N H^T]^T \, (O_N^T O_N)^{-1} O^T \quad .$$

Such an $\bar{O}$ can be partitioned in (N-1) blocks so that

$$\bar{O} = [P_0, P_1, \ldots, P_{N-1}]$$

where the P_i;s are pxp matrices over B. Therefore, by construction

$$[\hat{S}^N H^T]^T = \bar{O} O_N = [P_0, P_1, \ldots, P_{N-1}] \begin{bmatrix} H \\ (\hat{S}H^T)^T \\ \vdots \\ (\hat{S}^{N-1}H^T)^T \end{bmatrix} .$$

That is,

$$[\hat{S}^N H^T]^T = \sum_{i=0}^{N-1} P_i (\hat{S}^i H^T)^T \quad . \tag{4.16}$$

But, by hypothesis $\Sigma = (D, E, H)$ is a realization of f_N; therefore, from proposition (4.8)

$$(4.17) \qquad W_{ij} = (\hat{S}^{i-1}H^T)^T(S^{j-1}E) \quad , \quad \text{for } i,j \in Z^+ \quad .$$

In particular

$$W_{N+1,j} = (\hat{S}^N H^T)^T(S^{j-1}E) \quad , \quad j \in Z^+ \quad .$$

Next, we use ((4.16) to get

$$(4.18) \qquad W_{N+1,j} = \sum_{i=0}^{N-1} P_i(\hat{S}^i H^T)^T(S^{j-1}E) \quad , \quad j \in Z^+$$

and by substituting (4.17) into the right-hand side of (4.18) we have

$$W_{N+1,j} = \sum_{i=0}^{N-1} P_i W_{i+1,j} \quad \text{for all} \quad i \in Z^+ \quad .$$

Hence, the unit-pulse response matrix of f_W is recurrent over B.

(Sufficiency) Let us suppose that for f_W the recurrent relation in (4.15) holds for some pxp matrices P_0, P_1, ..., P_{q-1} over B. Consider the following pq-dimensional system $\Sigma = (D, E, H)$ over B

$$D = \begin{bmatrix} 0_p & I_p & 0_p & \cdots & 0_p & 0_p \\ 0_p & 0_p & I_p & \cdots & 0_p & 0_p \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0_p & 0_p & 0_p & \cdots & 0_p & I_p \\ \sigma P_0 & \sigma P_1 & \sigma P_2 & \cdots & \sigma P_{q-2} & \sigma P_{q-1} \end{bmatrix} , \quad E = \begin{bmatrix} W_{11} \\ W_{21} \\ \vdots \\ W_{q-1,1} \\ W_{q,1} \end{bmatrix} , \quad H = \begin{bmatrix} I_p \\ 0_p \\ \vdots \\ 0_p \\ 0_p \end{bmatrix} ,$$

where

$$I_p := (\Delta(i-j))_{pxp} \; ; \; 0_p := (0)_{pxp} \quad .$$

Let us show that $\Sigma = (D, E, H)$ realizes f_W over B. To do so, let us proceed as follows.

We know that

$$S^j E = D(\sigma(S^{j-1}E)) \quad , \quad \text{for all } j \in Z^+ \quad ,$$

$$S^0 = I$$

Clearly then

$$S^0 E = (W_{11}^T, W_{21}^T, \ldots, W_{q,1}^T)^T \quad .$$

Also

$$SE = D(\sigma E) = \begin{bmatrix} \sigma W_{21} \\ \sigma W_{31} \\ \vdots \\ \sigma W_{q,1} \\ \sum_{i=0}^{q-1} (\sigma P_i)\sigma W_{i+1,1} \end{bmatrix} .$$

But note that

$$\sum_{i=0}^{q-1} (\sigma P_i)\sigma W_{i+1,1} = \sigma\left(\sum_{i=0}^{q-1} P_i W_{i+1,1}\right) = \sigma W_{q+1,1} \text{ (by hypothesis)} \quad .$$

Hence

$$SE = [\sigma W_{21}^T;\ \sigma W_{31}^T;\ \ldots;\ \sigma W_{q+1,1}^T]^T$$

and by using lemma (4.13) we get

$$SE = [W_{12}^T;\ W_{22}^T;\ \ldots;\ W_{q,2}^T]^T \quad .$$

By following the same line of reasoning as in above, and by induction on j, it is easy to show that

$$(S^{j-1}E) = [W_{ij}^T;\ W_{2j}^T;\ \ldots;\ W_{q,j}^T] \text{ for all } j \in Z^+ \quad .$$

Therefore,

$$HS^{j-1}E = [I_p;\ 0_p;\ \ldots;\ 0_p] \begin{bmatrix} W_{1j} \\ W_{2j} \\ \vdots \\ W_{q,j} \end{bmatrix} = W_{1j} \text{ for all } j \in Z^+ \quad ,$$

and by proposition (4.8), we conclude that Σ realizes f_W.

Finally, we prove that $\Sigma = (D, E, H)$ is observable in q-steps at all times. So, let us construct $Q_{pq}^T = [H^T, \hat{S}H^T, \ldots, \hat{S}^{pq-1}H^T]$, where

$$\hat{S}^i H^T = \sigma^{-1}(\mathcal{D}^T(\hat{S}^{i-1} H^T)) \quad , \quad i \in Z^+$$

$\hat{S}^0$ = identity operator

Obviously,

$$\hat{S}^i H^T = \begin{cases} [0_p, 0_p, \ldots, \underset{\uparrow}{I_p}, \ldots, 0_p]^T \ , \text{ for } i \leq q \\ \qquad\qquad\quad i^{th} \text{ block entry} \\ 0 \qquad\qquad\qquad\qquad , \text{ for } i > q \end{cases} .$$

Therefore

$$0_q = (\Delta(i-j))_{pqxpq} = I_{pqxpq}$$

which clearly has rank pq at each $k \in Z$; that is, Σ is observable.

□

By duality we obtain the following result on the existence of reachable realizations $\Sigma = (D, E, H)$ over B of f_W.

(4.19) THEOREM: A response map $f_W : B_+^m \to B_+^p$ is realizable by a reachable system $\Sigma = (D, E, H)$ over B (i.e., rank $[E, SE, \ldots, S^{N-1}E](k) = \dim\Sigma$ at each $k \in Z$ for some $N \in Z^+$) if and only if there exist mxm matrices $Q_0, Q_1, \ldots, Q_{s-1}$ over B such that

$$W_{i,s+1} = \sum_{j=0}^{s-1} W_{i,j+1} Q_j \quad . \tag{4.20}$$

PROOF: Given the i/o map $f_W : B_+^m \to B_+^p : u \to f_W(u) : k \to \sum_{l=-\infty}^{k} W(k, l) u(l)$, consider its dual $f_{\hat{W}} : B_-^p \to B_-^m$ with unit-pulse response $\hat{W}(k, l) = W^T(l, k)$ for all $(l, k) \in CP$ (see (2.3.24)).

Define $\hat{W}_{ij}(k) := \hat{W}(k-i, k+j-1)$ for each $i,j \in Z^+$, each $k \in Z^+$, and note the following relationship

$$\hat{W}_{ij} = W_{ji}^T \quad . \tag{4.21}$$

Suppose f_W is recurrent over B in the sense of (4.20); that is,

$$W_{i,s+1} = \sum_{j=0}^{s-1} W_{i,j+1} Q_j \quad , \; i \in Z^+ \quad .$$

By taking transpose at both sides of the above equality, we have

$$\hat{W}_{s+1,i} = \sum_{j=0}^{s-1} Q_j^T \hat{W}_{j+1,i} \quad , \quad i \in Z^+ \quad .$$

By theorem (4.18), $f_{\hat{W}}$ has an ms-dimensional realization over B, $\Sigma = (D, E, H)$ which is observable. Therefore, by proposition (2.3.26) the dual system $\hat{\Sigma} = (\hat{D}, \hat{E}, \hat{H})$ (see definition (2.3.8)) is an ms-dimensional realization over B of f_W. Furthermore, it follows by duality that $\hat{\Sigma}$ is reachable. Hence sufficiency has just been shown.

Conversely, assume that f_W has an n-dimensional reachable realization $\Sigma = (D, E, H)$ over B. By duality, $f_{\hat{W}}$ is realizable over B by the n-dimensional observable system $\hat{\Sigma} = (\hat{D}, \hat{E}, \hat{H})$. And from theorem (4.18), there exist mxm matrices $P_0, P_1, \ldots, P_{q-1}$ over B such that

$$(4.22) \qquad \hat{W}_{q+1,j} = \sum_{i=0}^{q-1} P_i \hat{W}_{i+1,j} \quad , \quad j \in Z^+ \quad .$$

Take transpose at both sides of (4.22), and use (4.21) to get

$$(4.23) \qquad W_{j,q+1} = \sum_{i=0}^{q-1} W_{j,i+1} P_i^T \quad \text{for each } j \in Z^+ \quad .$$

Finally by making the following substitutions in (4.23): $i \to j$, $j \to i$, $q \to s$, $P_i^T \to Q_i$ we have

$$W_{i,s+1} = \sum_{j=0}^{s-1} W_{i,j+1} Q_j \quad , \quad \text{for all } i \in Z^+ \quad .$$

That is, f_W is recurrent over B.

□

(4.24) REMARK. Several observations about theorems (4.18) and (4.19) are worthwhile to make, among them:

(i) They are the discrete-time counterpart of similar results obtained by KAMEN [1979] for analytic time-varying systems. This is due to the fact that discrete time systems can be studied in terms of a semilinear transformation which is the

discrete-time counterpart to the differential transformation used in continuous-time systems.

(ii) When f_W has a time-invariant realization both criteria for realizability reduce into (4.12), so they can be considered as a generalization of such a result.

(iii) The realization $\Sigma = (D, E, H)$ of f_W used in the proof of theorem (4.18) is not minimal in general as it can been seen in the counterexample below.

(4.25) COUNTEREXAMPLE. Consider $f_W : B_+ \to B_+$ with unit-pulse response matrix $W(\cdot, \cdot)$ given by

$$W(k, l) = (\sin \tfrac{\pi}{2} k) \cos \tfrac{\pi}{2} (l + 1) \quad , \quad (k, l) \in CP \quad .$$

It can be shown that the Hankel matrix associated with f_W is given by

$$B_W(k) = \begin{bmatrix} 0 & \sin^2 \frac{\pi}{2} k & 0 & -\sin^2 \frac{\pi}{2} k & \dots \\ \cos^2 \frac{\pi}{2} k & 0 & -\cos^2 \frac{\pi}{2} k & 0 & \dots \\ 0 & -\sin^2 \frac{\pi}{2} k & 0 & \sin^2 \frac{\pi}{2} k & \dots \\ -\cos^2 \frac{\pi}{2} k & 0 & \cos^2 \frac{\pi}{2} k & 0 & \dots \\ 0 & +\sin^2 \frac{\pi}{2} k & 0 & -\sin^2 \frac{\pi}{2} k & \dots \\ \cos^2 \frac{\pi}{2} k & 0 & -\cos^2 \frac{\pi}{2} k & 0 & \dots \\ 0 & -\sin^2 \frac{\pi}{2} k & 0 & \sin^2 \frac{\pi}{2} \kappa & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \quad k \in Z.$$

Hence rank $B_W(k) = 1$ for each $k \in Z$ (note that for this i/o map B_W has constant rank!). Then by theorem (3.2.15), one is the dimension of a minimal realization of f_W; in fact, $\tilde{\Sigma} = (1, \cos \frac{\pi}{2} k, \sin \frac{\pi}{2} k)$ is a minimal state model for f_W.

On the other hand, it is not hard to see that

$$W_{3,j}(k) = (-1)W_{1,j}(k) + (0)W_{2,j}(k) \quad \text{for each } j \in Z^+, \text{ each } k \in Z \quad .$$

Hence, $s = 2$, $p_0 = -1$, $p_1 = 0$; and $\Sigma = (D, e, h)$ given by

$$D(k) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ; \quad e(k) = \begin{bmatrix} 0 \\ \cos^2 \frac{\pi}{2} k \end{bmatrix} ; \quad h(k) = [1 \quad 0]$$

is an observable realization of f_W (see proof of theorem (3.2.15)). However, it is not minimal; even though $s = 2$ is the smallest positive integer for which f_W is recurrent in the sense of (4.15). Furthermore, it is not even a minimal observable realization of f_W (that is, an observable realization of f_W with smallest dimension) because it can be shown that the minimal realization $\tilde{\Sigma}$ given above is also observable.

CHAPTER FIVE
GLOBAL REALIZABILITY CRITERIA BASED ON THE HANKEL MATRIX

Most of the existing realization algorithms for both time-invariant systems and time-varying continuous-time systems (see Chapter Two) are based on the notion of the Hankel matrix associated with a given input/output map. For example, in the theory of realization for time-invariant response maps over a ring, it has been shown that realizability is equivalent to finiteness of the modules generated by the elementary rows (or columns) of the Hankel matrix (see ROUCHALEAU [1972] for details). Furthermore, such a condition is also true for realizability of real analytic time-varying integral operators as it was shown in KAMEN [1979]. Therefore, it seems natural then to develop a similar approach for discrete-time varying system.

Given a linear i/o map $f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p$ with unit pulse response matrix $W(\cdot, \cdot)$, we have defined as its (generalized) Hankel matrix $\mathcal{B}_W(\cdot)$ the following infinite matrix (see definition (3.2.12))

$$\mathcal{B}_W := \begin{bmatrix} W_{11} & W_{12} & W_{13} & \cdots \\ W_{21} & W_{22} & W_{23} & \cdots \\ W_{31} & W_{32} & W_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where

$$W_{ij} : Z \to K^{pxm} : k \to W(k+i-1, k-j) \quad \text{for each } i,j \in Z^+ \quad . \tag{5.2}$$

We have not expressed the time-dependence of $B_W(\cdot)$ in (5.1) to emphasize the fact that its entries are over B.

Let R_i denote the i^{th}-block row of B_W; that is, $R_i := [W_{i1}, W_{i2}, \ldots]$ where the W_{ij}'s are given by (5.2). Denote by B^{pxp} the (non-commutative) ring of all (pxp)-matrices over B, and define X_W^R as the set of all finite-linear combinations $\sum_i P_i R_i$ with $P_i \in B^{pxp}$, where $P_i R_i := [P_i W_{i1}, P_i W_{i2}, \ldots]$. It is rather easy to show that X_W^R becomes a left-module over the ring B^{pxp} with componentwise addition and with scalar multiplication defined by $PR_i := [PW_{i1}, PW_{i2}, \ldots]$.

Also, let C_i be the i^{th}-block column of B_W, i.e., $C_i := [W_{1i}^T, W_{2i}^T, \ldots]$ for each $i \in Z^+$. Define X_W^C as the set of all finite linear combinations of the form $\sum_i C_i Q_i$ with $Q_i \in B^{mxm}$, and $C_i Q_i := [(W_{1i}Q_i)^T, (W_{2i}Q_i)^T, \ldots]$. With addition and scalar multiplication defined componentwise, it is easy to see that X_W^C becomes a right-module over B^{mxm}.

Given a left-(right) module M over a ring F, we shall denote the F-submodule generated by elements $m_1, m_2, \ldots, m_q$ in M by ${}_F\langle m_1, m_2, \ldots, m_q\rangle$ ($\langle m_1, m_2, \ldots, m_q\rangle_F$).

(5.3) PROPOSITION: *There exist pxp matrices $P_0, P_1, \ldots, P_{q-1}$ over B satisfying*

$$W_{q+1,j} = \sum_{i=0}^{q-1} P_i W_{i+1,j} \quad , \quad j \in Z^+ \quad ,$$

<u>if and only if</u> X_W^R <u>is finitely generated as a left-module over</u> B^{pxp} <u>with generating set</u> $\{R_1, R_2, \ldots, R_q\}$.

<u>PROOF</u>: Assume that X_W^R is generated by the blocks rows $R_1, R_2, \ldots, R_{q-1}$ of B_W; that is

$$B^{pxp}\langle R_1, R_2, \ldots, R_q\rangle = X_W^R \quad .$$

Hence, for some matrices $P_0, P_1, \ldots, P_{q-1}$ in B^{pxp}, we have that

$$R_{q+1} = \sum_{i=0}^{q-1} P_i R_{i+1} \quad .$$

That is,

$$[W_{q+1,1}; W_{q+1,2}; \ldots; W_{q+1,j}, \ldots] =$$

$$[\sum_{i=0}^{q-1} P_i W_{i+1,1}; \sum_{i=0}^{q-1} P_i W_{i+1,2}; \ldots; \sum_{i=0}^{q-1} P_i W_{i+1,j}; \ldots] \quad .$$

By comparing component by component we get

$$W_{q+1,j} = \sum_{i=0}^{q-1} P_i W_{i+1,j} \quad , \quad \text{for all } j \in Z^+ \quad .$$

Conversely, let us assume that there exist $\rho x \rho$ matrices P_0, P_1, ..., P_{q-1} over B such that

$$W_{q+1,j} = \sum_{i=0}^{q-1} P_i W_{i+1,j} \quad \text{for each } j \text{ in } Z^+ \quad .$$

By theorem (4.14) the system $\Sigma = (D, E, H)$ given by

$$D = \begin{bmatrix} 0_\rho & I_\rho & 0_\rho & \cdots & 0_\rho & 0_\rho \\ 0_\rho & 0_\rho & I_\rho & \cdots & 0_\rho & 0_\rho \\ \vdots & \vdots & \vdots & & & \\ 0_\rho & 0_\rho & 0_\rho & \cdots & 0_\rho & I_\rho \\ \sigma P_0 & \sigma P_1 & \sigma P_2 & \cdots & \sigma P_{q-2} & \sigma P_{q-1} \end{bmatrix}, \quad E = \begin{bmatrix} W_{11} \\ W_{21} \\ \vdots \\ \\ W_{q,1} \end{bmatrix}, \quad H = [I_\rho, 0_\rho, \ldots, 0_\rho]$$

is a ρq-dimensional observable realization of f_W. We also proved that

$$[\hat{S}^{i-1}H^T]^T = [0_\rho; 0_\rho; \ldots; 0_\rho; \underset{\substack{\uparrow \\ i^{th}\text{-position}}}{I_\rho}; 0_\rho; \ldots; 0_\rho] \quad \text{for} \quad i = 1, 2, \ldots q \ .$$

Therefore, for any fixed non-negative integer μ, there exist $\rho x \rho$ matrices $\Phi_{0\mu}$, $\Phi_{1\mu}$, ..., $\Phi_{q-1,\mu}$ over B such that

$$[\hat{S}^{q+\mu}H^T]^T = \sum_{i=0}^{q-1} \Phi_{i,\mu}(\hat{S}^i H^T)^T \quad .$$

By proposition (4.8) we know that

$$W_{q+\mu+1,j} = (\hat{S}^{q+\mu}H^T)^T(S^{j-1}E) \quad , \quad \text{for each } j \in Z^+ \quad .$$

Hence

$$W_{q+\mu+1,j} = \sum_{i=0}^{q-1} \psi_{i,\mu} (\hat{S}^i H^T)(S^{j-1}E) \quad , \quad j \in Z^+ \quad .$$

Using again proposition (4.8) we get

$$W_{q+\mu+1,j} = \sum_{i=0}^{q-1} \psi_{i,\mu} W_{i+1,j} \quad \text{for each } j \in Z^+, \text{ each } \mu \in Z^+ \cup \{0\} \quad .$$

Finally, from the definition of $R_{q+\mu+1}$ (recall that $R_i = [W_{i1}, W_{i2}, \ldots]$) we get

$$R_{q+\mu+1} = \sum_{i=0}^{q-1} \psi_{i,\mu} R_{i+1} \quad , \quad \text{for all } \mu = 0, 1, 2, \ldots$$

relation which says that X_W^R is finitely generated as a left-module over the ring $B^{p\times p}$ with $\{R_1, R_2, \ldots R_q\}$ as a generating set

□

(5.4) COROLLARY: A linear i/o map $f_W : B_+^m \to B_+^p$ is realizable over B by an observable system $\Sigma = (D,E,H)$ if, and only if X_W^R is finitely generated as a left-module over $B^{p\times p}$.

PROOF: Just combine theorem (4.14) and proposition (5.3)

□

(5.5) PROPOSITION: There exist mxm matrices Q_0, Q_1, . . . , Q_{s-1} over B such that the unit-pulse response matrix W of f_W satisfies

$$W_{i,s+1} = \sum_{j=0}^{s-1} W_{i,j+1} Q_j \quad \text{for each } i \in Z^+$$

if and only if X_W^C is finitely-generated as a right-module over B^{mxm} with generating set $\{C_1, C_2, \ldots, C_q\}$.

PROOF: Just use the dual arguments in the proof of proposition (5.3). □

(5.6) COROLLARY: An i/o map $f_W : B_+^m \to B_+^p$ with unit-pulse response matrix $W(\cdot, \cdot)$ is realizable over B by a reachable system $\Sigma = (D, E, H)$ if, and only if, X_W^C in finitely generated as a right-module over B^{mxm}.

PROOF: Just use proposition (5.5) and theorem (4.19). □

Let $r_i(c_i)$ denote the i^{th}-elementary row (column) of the generalized Hankel matrix B_W associated with an i/o map f_W. Denote by $X_W^r(X_W^c)$ the A-module generated by the r_i's (c_i's) with addition and scalar multiplication defined componentwise. Note that we have not used the distinction between left or right in the above definition for multiplication on B is commutative. Furthermore, it is important to note the fact that, in general, X_W^r and X_W^c are not torsion-free modules over B since it can happen that there is a non-zero element m

ε X_W^r (X_W^c) for which there exists a non-zero element a in B with the property am = 0, i.e., m is a torsion element.

Observe that the B-module $X_W^r(X_W^c)$ is finitely generated over B if, and only if the B^{pxp}-left module (B^{mxm}-right module) $X_W^R(X_W^C)$ is finitely generated. Therefore, the following result is easily deduced from corollaries (5.4) and (5.6).

(5.7) THEOREM: A response map $f_W : B_+^m \to B_+^p$ is realizable over B by a reachable (observable) system $\Sigma = (D, E, H)$ if and only if $X_W^c(X_W^r)$ is finitely generated as a module over B.

An interesting question is whether or not condition (4.15) or condition (4.20) is necessary for realizability. It was shown in KAMEN [1979] that for analytic input/output maps f_W, such is not the case in general. However, for discrete-time varying response maps it is true as we now show. By the way, this fact is another significant difference between the theory of realization for continuous-time systems and the corresponding one for discrete-time systems.

(5.8) THEOREM: Given a linear i/o map $f_W : B_+^m \to B_+^p$ with unit-pulse response matrix W. Let $X_W^r(X_W^c)$ be the B-module generated by the elementary rows (columns) of the Hankel matrix $\mathbf{B}_W$ associated with f_W. Then the following are equivalent

(i) f_W is realizable,

(ii) X_W^r is a finitely-generated B-module,

(iii) X_W^c is a finitely-generated B-module.

PROOF: Assume that $f_W : B_+^m \to B_+^p$ is realizable, then by theorem (3.2.15), there exists a non-negative integer n such that

$$\text{rank } B_W(k) \leq n \quad \text{for all} \quad k \in Z \quad .$$

Therefore, for any fixed positive integer t we have that

$$\text{rank } B_W(t) = n_t \leq n \quad .$$

Such a relation says that the K-vector space generated by the elementary columns of $B_W(t)$, $X_W^c(t)$, has dimension n_t. Therefore, there exist vectors $b_1^{(t)}, \ldots, b_{n_t}^{(t)}$ in $X_W^c(t)$ which form a basis for such vector space. Also we known that $n_t \leq n$, so we pick vectors $b_{n_t+1}^{(t)}, \ldots, b_n^{(t)}$ in $X_W^c(t)$, which can be completely arbitrary, and form a set $\{b_1^{(t)}, b_2^{(t)}, \ldots, b_n^{(t)}\} := B(t)$. Clearly, B(t) is a generating set for $X_W^c(t)$ though it is not a basis set.

Finally define the following vectors in X_W^c (the B-module spanned by the elementary columns of B_W)

$$b_i(t) := b_i^{(t)} \quad \text{for each} \quad t \in Z \quad .$$

Obviously the set $B := \{b_1, b_2, \ldots, b_n\}$ generates X_W^c as an B-module. That is, X_W^c is finitely generated.

By duality, we can also show that (i) implies (iii). The fact that (ii) or (iii) implies (i) is a consequence of theorem (5.7).

□

Note that the above is the discrete-time version of theorem (2.2.18) due to Kamen. However, there is a basic difference in that for realizable linear i/o maps $f_W : B_+^m \to B_+^p$ the B-modules X_W^r or X_W^c do not need to be free in general (tnat fact is studied in more depth in Chapter Six). Another important difference between analytic continuous-time systems and discrete-time systems is that for the former it is possible to have $X_W^c(X_W^r)$ finitely generated while $X_W^r(X_W^c)$ is not (see example 1 in KAMEN [1979]). But for discrete-time realizable i/o maps, both X_W^r and X_W^c must be finite modules by theorem (5.8).

CHAPTER SIX
FINITE CONSTANT RANK INPUT/OUTPUT MAPS

In the process of finding a state-variable representation $\Sigma = (D, E, H)$, if there exists one, for a linear system described by its response map $f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p$ we should always look for a Σ having minimal dimension. Now, recall that the input/output data generated by a realization Σ is determined only for those states (of Σ) which are both reachable and observable; hence, it is necessary to impose a further condition on Σ, and that is "Σ must be canonical". By proposition (2.1.14), we have that any minimal realization of a given time-invariant i/o map f_W must be canonical, and vice versa; therefore, both requirements are equivalent. For continuous time-varying integral operators f_W the situation is more complicated because minimality does not imply canonical in general (see SILVERMAN [1971]); although; the converse is always true.

In this section we study the relationship between minimal realizations and canonical ones of a discrete time-varying i/o maps; and we shall determine cases in which a canonical realization always exists. Furthermore, some comparisons between the theory of realization for continuous-time systems and the corresponding one for discrete-time systems will be made.

One of the most important classes of continuous-time systems is the class of systems which have constant rank (SILVERMAN and MEADOWS [1969]). Recall that a system $\Sigma = (F, G, H)$ over $A(J)$ is said to be of constant-rank if there exist positive integers α, β, n_r and n_0 such that

$$\text{rank } R_i(t) = n_r \quad \text{for all} \quad i \geq \alpha \quad \text{and all } t \in J$$

$$\text{rank } O_i(t) = n_0 \quad \text{for all } i \geq \beta \quad \text{and all } t \in J$$

where

$$R_i := [G, D_F(G), \ldots, D_F^{i-1}(G)] = i^{th}\text{-reachability matrix of } \Sigma$$

$$O_i := [H^T, D_F^{\hat{}}(H^T), \ldots, D_F^{\hat{}i-1}(H^T)]^T = i^{th}\text{-observability matrix of } \Sigma$$

(see (2.2.25) and (2.2.26)).

Given an integral operator $f_W : C^m(J) \to D^p(J)$, the following interesting facts are worthwhile to recall:

(6.0-a) (SILVERMAN [1971]). If f_W is realizable by a constant-rank system, then there exists a minimal realization which is also constant rank. In fact, any minimal realization is canonical.

(6.0-b) (KAMEN [1979]). An f_W is realizable by a constant-rank system iff there exist positive integers n and μ such that

$$\text{rank } B_W^{(i)}(t) = n \quad \text{for all} \quad i \geq \mu, \quad \text{and all } t \in J$$

(see theorem (2.2.29)).

(6.0-c) (KAMEN [1979]). The modules X_W^c and X_W^r are finite and free with rank X_W^c = rank $X_W^r = n$ if, and only if, f_W is realizable by a constant-rank system.

It is natural then to find the discrete-time counterpart for the above results. Due to the lack of continuity of the rank of a given matrix over $\mathcal{B}$, the following definition of constant rank system in the discrete framework is adopted:

(6.1) DEFINITION: Given a discrete time-varying system $\Sigma = (D, E, H)$ over $\mathcal{B}$, we shall say that Σ is a constant rank system if there exist positive integers α, β, n_r and n_o such that

$$\text{rank } R_i(k) = n_r \quad \text{for all } i > \alpha \text{ and all } k \in Z$$

$$\text{rank } O_i(k) = n_o \quad \text{for all } i > \beta \text{ and all } k \in Z$$

where

$$R_i := [E, SE, \ldots, S^{i-1}E] = i^{th}\text{-reachability matrix of } \Sigma,$$

$$O_i := [H^T, \hat{S}H^T, \ldots, \hat{S}^{i-1}H^T]^T = i^{th}\text{-observability matrix of } \Sigma .$$

(6.2) EXAMPLE: Let $\Sigma = (D, E, H)$ be given by

$$D(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E(k) = \begin{bmatrix} \Delta(k) & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}; \quad H^T(k) = \begin{bmatrix} \sin \frac{\pi}{2} k & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is not hard to show that for each $i \in Z^+$

$$R_i(k) = \begin{bmatrix} \Delta(k) & 1 & \Delta(k-1) & 1 & \Delta(k-2) & 1 & \ldots & \Delta(k-i) & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad k \in Z.$$

Clearly

$$\text{rank } R_i(k) = 2 \quad \text{for all, } i \geq 1 \text{ and all } k \in Z .$$

On the other hand

$$O_2(k) = \begin{bmatrix} \sin \frac{\pi}{2} k & 1 & 0 \\ 0 & 1 & 1 \\ \sin\pi(k+1)/2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad k \in Z \quad .$$

Therefore, rank $O_2(k) = 3$ for all $k \in Z$ (Σ is observable).

From definition (6.1), Σ is a constant rank system.

Given a linear i/o map $f_W : B_+^m \to B_+^p$ with unit-pulse response matrix W, let B_W be its (generalized) Hankel matrix (see definition (3.2.12)). For each $i \in Z^+$, associate with f_W the i^{th}-submatrix $B_W^{(i)}$ of B_W defined by

$$B_W^{(i)}(k) := \begin{bmatrix} W_{11}(k) & W_{12}(k) & \cdots & W_{12}(k) \\ W_{21}(k) & W_{22}(k) & \cdots & W_{2i}(k) \\ \vdots & \vdots & & \vdots \\ W_{i1}(k) & W_{i2}(k) & \cdots & W_{ii}(k) \end{bmatrix} \quad , \text{ for } k \in Z$$

where

$$W_{ij}(k) = W(k+i-1,k-j) \quad i,j \in Z^+ \, , \quad k \in Z \quad .$$

(6.4) PROPOSITION: A linear input/output map $f_W: B_+^m \to B_+^p$ is realizable by a finite-dimensional system $\Sigma = (U,E,H)$ over B if and only if for each positive integer i

$$B_W^{(i)} = O_i R_i \quad ,$$

where $O_i(R_i)$ is the i^{th}-observability (reachability) matrix of Σ (see (2.3.32))

PROOF: By proposition (4.8), f_W is realizable by Σ if and only if

$$W_{ij} = [\hat{S}^{i-1}H^T]^T[S^{j-1}E] \quad \text{for each } i,j \in Z^+ \quad .$$

Therefore, from (6.3) Σ is a realization of f_W if for each $i \in Z^+$

$$B_W^{(i)} = \begin{bmatrix} (\hat{S}^0H^T)^T(S^0E) & (\hat{S}^0H^T)^T(SE) & \cdots & (\hat{S}^0H^T)^T(S^{i-1}E) \\ (\hat{S}H^T)^T(S^0E) & (\hat{S}H^T)^T(SE) & & (\hat{S}H^T)^T(S^{i-1}E) \\ \vdots & \vdots & & \vdots \\ (\hat{S}^{i-1}H^T)^T(S^0E) & (\hat{S}^{i-1}H^T)^T(SE) & & (\hat{S}^{i-1}H^T)^T(S^{i-1}E) \end{bmatrix}$$

$$= \begin{bmatrix} (\hat{S}^{o}H^{T})^{T} \\ (\hat{S}H^{T})^{T} \\ \vdots \\ (\hat{S}^{i-1}H^{T})^{T} \end{bmatrix} \begin{bmatrix} S^{o}E & SE & \cdots & S^{i-1}E \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \mathbf{O}_i R_i$$

□

The preceding result will be used frequently in the sequel.

(6.5) DEFINITION: A linear input/output map $f_W: B_+^m \to B_+^p$ is said to have finite rank if there exist positive integers μ and n such that

(6.6) $\quad \text{rank } B_W^{(i)}(k) \leq n$ for all $i \geq \mu$ and all $k \in Z$

(6.7) THEOREM: A linear i/o map $f_W: B_+^m \to B_+^p$ has finite rank if and only if it is realizable.

PROOF: Assume that f_W is realizable over the causal-half plane CP. By theorem (3.2.15) there exists a finite positive integer n such that

(6.8) $\quad \text{rank } B_W(k) \leq n$ for each $k \in Z$

Since $B_W^{(i)}$ is a submatrix of B_W (see (6.3)), it must be true that

$$\text{rank } B_W^{(i)}(k) \leq \text{rank } B_W(k) \quad \text{for each } i \in Z^+ \text{ and each } k \in Z$$

By using (6.8) we have

$$\text{rank } B_W^{(i)}(k) \leq n \text{ for all } i \in Z^+, \text{ and all } k \in Z$$

Conversely, suppose that (6.6) holds. By taking into account that $B_W^{(i)} \to B_W$ as $i \to \infty$, we obtain that rank $B_W(k) \leq n$ for all $k \in Z$. Thus f_W is realizable (by theorem (3.2.15)) .

□

Next, we introduce a class of i/o maps f_W which have several nice properties as we shall see later.

(6.9) DEFINITION: An input/output map $f_W: B_+^m \to B_+^p$ is said to have finite constant rank if there exist positive integers μ, n such that

(6.10) $\text{rank } B_W^{(i)}(k) = n$ for all $i \geq \mu$ and all $k \in Z$.

At once we have the following result.

(6.11) PROPOSITION: Let $f_W: B_+^m \to B_+^p$ be a finite constant rank response map. Then the following statements are true

(i) f_W is realizable, and n (see (6.10)) is the dimension of a minimal realization ,

(ii) for each $\lambda \in Z$, f_W has an n-dimensional minimal realization on CP_λ (recall $CP_\lambda := \{(k,\ell) \in Z \times Z: k \geq \lambda > \ell\}$).

PROOF:

(i) from definition (6.5) and theorem (6.7), it follows that any i/o map f_W with finite constant rank is realizable. The last part of the statement is obtained by using theorem (3.2.15)

(ii) It follows directly from (i) and from theorem (3.2.9). (Here you need to recall that $C^{\lambda}(W) \equiv B_W(\lambda)$.)

□

(6.12) REMARK: In general, a realizable i/o map f_W need not be finite constant rank. For a counterexample see (3.2.17).

(6.13) PROPOSITION: Let $f_W: B_+^m \to B_+^p$ be a linear i/o map with finite constant rank. Then f_W is realizable by a constant rank system $\Sigma = (D,E,H)$ over B (see definition (6.1)).

PROOF: From the definition of a finite constant rank response map f_W, we have that there exist μ, $n \in Z^+$ so that

(6.14) $\quad \text{rank } B_W^{(i)}(k) = n \text{ for all } i > \mu,\ k \in Z.$

By proposition (6.11), a minimal realization $\Sigma = (D,E,H)$ of f_W must have dimension n. Now, let $R_i(\mathcal{O}_i)$ denote the i^{th}-reachability (observability) matrix associated with Σ. Then, it must be clear from (2.3.32) that for each $i \in Z^+$

(6.15) $\quad \text{rank } R_i(k) \leq n, \text{ for all } k \in Z$

(6.16) rank $O_i(k) \leq n$, for all $k \in Z$.

Since Σ realizes f_W, from proposition (6.9) it is known that for each $i \in Z^+$

(6.17) $B_W^{(i)} = O_i(k)R_i(k)$, $k \in Z$

Apply Sylvester's law to (6.17) to obtain

(6.18) rank $B_W^{(i)} \leq \min \{\text{rank } O_i(k), \text{rank } R_i(k)\}$, $k \in Z$

Take $i \geq \mu$, and use (6.14) to conclude from (6.18) that

(6.19) $n \leq$ rank $O_i(k)$, for all $k \in Z$

(6.20) $n \leq$ rank $R_i(k)$, for all $k \in Z$.

Finally combine (6.15) with (6.20) and (6.16) with (6.19) to conclude that

rank $R_i(k) = n$ for all $i \geq \mu$, $k \in Z$

rank $O_i(k) = n$ for all $i \geq \mu$, $k \in Z$.

Therefore Σ is a constant rank realization of f_W (note that Σ is canonical).

Unfortunately, the converse of proposition (6.13) is not true in general. A counterexample is given below.

(6.21) COUNTEREXAMPLE: Consider the scalar system $\Sigma = (D,e,h)$ given by

$$D = 0;\ e(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix};\ h(k) = (h_1(k),\ h_2(k))$$

where

$$h_1(k) = h_2(k) = 1 \text{ for all } k \neq 0$$

$$h_1(0) = -h_2(0) = 1$$

It is not hard to see that

$$\text{rank } R_i(k) = 1 \text{ for all } i \geq 1, \text{ and all } k \in Z$$

$$\text{rank } O_i(k) = 1 \text{ for all } i \geq 1, \text{ and for all } k \in Z$$

That is, Σ is a constant-rank system which, of course, realizes its own i/o map f_Σ. However, it can be shown that the generalized Hankel matrix $\mathcal{H}_W$ associated with f_Σ is given by

$$B_W(k) = \begin{bmatrix} h_1(k) + h_2(k) & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}, \quad k \in Z$$

As a result, we have that

$$\text{rank } B_W^{(i)}(k) = \text{rank } B_W^{(1)}(k) = \begin{cases} 1 \text{ for } k \neq 0 \\ \\ 0 \text{ for } k = 0 \end{cases}$$

for all $i > 1$.

Hance f_Σ: $B_+ \to B_+$ is not a finite constant rank i/o map; although it has a constant rank realization.

The above example shows a basic difference between the theory of realization for discrete i/o maps and the corresponding one for analytic continuous-time integral operators. The difference is that in the latter (see (6.0-b)) the converse of the above proposition is always true.

(6.22) COROLLARY: Any minimal realization $\Sigma = (D,E,H)$ of a finite constant rank i/o map f_W: $B_+^m \to B_+^p$ is canonical.

PROOF: It follows from the proof of proposition (6.13).

The above result shows us an interesting characteristic of any finite constant rank response map f_W which is that "minimal" implies "canonical". However, this is not true for arbitrary realizable i/o maps as we now show with the following

(6.23) COUNTEREXAMPLE (WEISS [1972]) Consider the i/o map $f_W: B_+ \to B_+$ whose unit-pulse response is given by

$$W(k,\ell) = (\Delta(k-5) + (\Delta(k-6))(\Delta(\ell-3) + \Delta(\ell-4)) \text{ for all } (k,\ell) \in CP,$$

where

$$\Delta(j) = \begin{cases} 1, \text{ if } j = 0 \quad ; \\ 0, \text{ otherwise} \quad . \end{cases}$$

Weiss gives the following minimal realization of f_W

$$d \equiv 1;\ e(k) = \Delta(k-4) + \Delta(k-5);\ h(k) = \Delta(k-5) + \Delta(k-6) \quad .$$

For such systems

$$\text{rank } R_i(k) = \begin{cases} 1, & k = 4, 5, 6 \ldots (5 + i - 1) \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{rank } O_i(k) = \begin{cases} 1, & k = (5 - i + 1), \ldots 5, 6, \\ 0, & \text{otherwise.} \end{cases}$$

That is, Σ is neither reachable nor observable.

We have seen then that Σ is a minimal realization but is not canonical. This fact was not observed by Weiss.

(6.24) PROPOSITION: Let $\Sigma = (D,E,H)$ be an n-dimensional canonical realization of a given linear i/o map $f_W: \mathcal{B}_+^m \to \mathcal{B}_+^p$. Then f_W has finite constant rank and Σ is a minimal realization.

PROOF: Σ canonical implies that there exist positive integers α, β such that

(6.25) $\qquad \text{rank } R_i(k) = n \text{ for all } i \geq \alpha, \text{ and all } k \in Z,$

(6.26) $\qquad \text{rank } O_i(k) = n \text{ for all } i \geq \beta, \text{ and all } k \in Z.$

Since Σ realizes f_W by proposition (6.4) we have that for each $i \in Z^+$

(6.27) $$B_W^{(i)}(k) = O_i(k)R_i(k) \text{ at every } k \in Z.$$

Apply Sylvester's law to the relation (6.27) to get

(6.28) $$\text{rank } B_W^{(i)}(k) \leq \min \{\text{rank } O_i(k), \text{rank } R_i(k)\}, \; i \in Z^+, \; k \in Z$$

By taking $i \geq \mu := \max(\alpha,\beta)$ and by using (6.25) and (6.26) we get from (6.28) that

(6.29) $$\text{rank } B_W^{(i)}(k) \leq n \text{ for each } i \geq \mu, \; k \in Z.$$

On the other hand, (6.24) implies that for each $i \geq \alpha$, and for each $k \in Z$ the matrix $R_i(k)$ has a right inverse $\tilde{R}_i(k)$. Therefore, by premultiplying both sides of (6.26) by $\tilde{R}_i(k)$ we have

(6.30) $$B_W^{(i)}(k)\tilde{R}_i(k) = O_i(k) \quad i \geq \mu, \; k \in Z$$

Apply Sylvester's law to (6.30) to obtain

$$\text{rank } B_W^{(i)}(k) \geq n \quad \text{for all } i \geq \mu, \; k \in Z$$

which together with (6.29) says that

$$\text{rank } B_W^{(i)}(k) = n \quad \text{for all } i \geq \mu, \; k \in Z.$$

Hence, f_W has finite constant rank and by proposition (6.11) Σ is a minimal realization.

□

Now, by combining corollary (6.22) with the above proposition we can show the following result.

(6.31) THEOREM: Let $f_W: \mathcal{B}_+^m \to \mathcal{B}_+^p$ be a linear input/output map with finite constant rank. A realization $\Sigma = (D,E,H)$ of f_W is minimal if and only if it is canonical.

The interesting feature of this result is that it is the discrete-time counterpart of a result due to SILVERMAN [1971] (see also theorem (2.2.28)) for continuous time-varying systems. Such a characteristic of finite constant rank i/o maps f_W is of basic importance in this project because it will allow us to find concrete realizations of f_W (see Chapters Seven and Eight).

To really appreciate the importance of linear i/o maps $f_W: \mathcal{B}_+^m \to \mathcal{B}_+^p$ with finite-constant rank, let us show the following striking result.

(6.32) THEOREM: Let $f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p$ be a response map with unit-pulse response matrix $W(\cdot, \cdot)$. Then f_W has a canonical realization if and only if it has finite constant rank.

PROOF: If f_W has a canonical realization, then it has finite-constant rank by proposition (6.24). On the other hand, if f_W has finite-rank, then it has a canonical realization by corollary (6.22).

□

Therefore, the preceding result tells us that for the only kind of discrete time-varying i/o maps that there exists a canonical realization is those with finite constant rank.

(6.33) PROPOSITION: Let $f_W : B_+^m \to B_+^p$ be an i/o map with finite constant rank n, then the B-modules X_W^r and X_W^c are both finite and free with rank n.

PROOF: Since any i/o map f_W with constant rank is realizable, by theorem (5,8) we get that both B-modules X_W^r and X_W^c are finitely generated. So we need to show only that they are free with rank n.

By definition there exists a positive integer μ such that

(6.34) $\quad \text{rank}B_W^{(i)}(k) = n \quad$ for all $\quad i \geq \mu, \quad$ and all $\quad k \in Z \quad .$

Next, define the following column-truncated Hankel matrix for each i in Z^+:

$$ctB_W^{(i)} := \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1i} \\ W_{21} & W_{22} & \cdots & W_{2i} \\ W_{31} & W_{32} & \cdots & W_{3i} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

where the prefix "ct" stands for column truncated.

Because of (6.34) it must be clear that for each $i \geq \mu$

$$\text{rank}\mathbf{B}_W^{(i)}(k) = \text{rank}(\text{ct}\mathbf{B}_W^{(i)}(k)) = \text{rank}\mathbf{B}_W(k), \quad k \in Z \quad . \tag{6.35}$$

Since X_W^c is finitely generated by the first μm-elementary columns of $\mathbf{B}_W$ which are the elementary columns of $\text{ct}\mathbf{B}_W^{(\mu)}$, we must have that

$$X_W^c = \text{B-module generated by the columns of } \text{ct}\mathbf{B}_W^{(\mu)} \quad . \tag{6.36}$$

From (6.35) we obtain that

$$\text{rank}(\text{ct}\mathbf{B}_W^{(\mu)}(k)) = n \quad \text{for each } k \in Z \quad .$$

So, for each k in Z, there exist n columns $c_{i_1}(k)$, ..., $c_{i_n}(k)$ of $\text{ct}\mathbf{B}_W^{(\mu)}(k)$ which are a basis for $X_W^c(k)$ = K-vector space spanned by the elementary columns of $\text{ct}\mathbf{B}_W^{(\mu)}(k)$.

Next, denote by B^∞ the B-module consisting of all infinite column vectors over B (addition and scalar multiplication are defined componentwise). Define the following elements in B^∞

$$b_j(k) := c_{i_j}(k) \quad \text{for each } j \text{ in } \{1, 2, \ldots, n\}, \text{ all } k \in Z \quad .$$

From the very definition of the b_j's, it is easy to see that they are K-linear independent at each $k \in Z$. Therefore, by lemma (A.1.5), b_1, b_2, ..., b_n are B-linearly independent. Furthermore, it must be clear that they are also elements of X_W^c.

From (6.36) it follows that any x in χ_W^C can be written as

$$x = \sum_{i=1}^{m\mu} a_i c_i \quad , \quad a_i\text{'s in } \mathbf{B},$$

but note that for each $i \in \{1, 2, \ldots, m\}$ and for each $k \in Z$

$$c_i(k) = \sum_{j=1}^{n} \alpha_{ji}(k) b_j(k) \quad \text{for some } \alpha_{ji}(k)\text{'s in } K \quad .$$

Hence, it must be clear that

$$x(k) = \sum_{j=1}^{n} \left(\sum_{i=1}^{\mu m} a_i(k)\ \alpha_{ji}(k) \right) b_j(k) \quad \text{for each } k \in Z \quad .$$

That is,

$$x = \sum_{j=1}^{n} x_j b_j \quad ,$$

where $x_j(k) = \sum_{i=1}^{\mu m} a_i(k)\alpha_{ji}(k)$, for each $k \in Z$.

Since x in χ_W^C was completely arbitrary we have that

$$\chi_W^C = \langle b_1, b_2, \ldots, b_n \rangle_B \quad .$$

We have thus shown that $b_1, \ldots, b_n$ are a basis for the B-module χ_W^C. Therefore, χ_W^C is free with rank n.

By the same line of reasoning as above and by working with B_W^T, it is possible to show that the B-module X_W^r is also free with rank equal to n.

□

The next natural step is to see whether or not the converse of the above proposition is also true.

(6.37) PROPOSITION: *Let* $f_W : B_+^m \to B_+^p$ *be a linear i/o map, and consider the B-modules* X_W^r *and* X_W^c *associated with its generalized Hankel matrix* $B_W(\cdot)$. *If* X_W^r *and* X_W^c *are both finitely generated and free with rank* n, *then there exists a positive integer* μ *such that*

$$\text{rank} B_W^{(i)}(k) = n \text{ for all } i > \mu, \text{ and all } k \in Z \ ;$$

that is, f_W *has finite constant rank.*

PROOF: By hypothesis both X_W^r and X_W^c are finitely generated. It must be clear then that both X_W^R and X_W^C are also finitely generated. And from propositions (5.3) and (5.5), there exist positive integers q and s such that

(a) $\{R_1, \ldots, R_q\}$ is a generating set of X_W^R as right module over $B^{\mu xp}$

(b) $\{C_1, \ldots, C_s\}$ is a generating set of X_W^C as a left module over $B^{mx\mu}$

where $R_i(C_i)$ is the i^{th}-block row (column) of B_W.

With $\mu := \max(q, s)$ we have that X_W is generated by $R_1, \ldots, R_\mu$ while X_W^C is generated by $C_1, C_2, \ldots, C_\mu$. Therefore

$$\text{rank}\mathbf{B}_W^{(i)}(k) = \text{rank}\mathbf{B}_W^{(\mu)}(k) \quad \text{for all } i > \mu \text{ , and all } k \in Z \text{ .}$$

To complete the proof, then, we need only to show that

$$\text{rank } \mathbf{B}_W^{(\mu)}(k) = n \text{ for all } k \in Z.$$

By hypothesis X_W^C is the free with rank n. Then by definition, X_W^C admits a basis set $\{b_1, b_2, \ldots, b_n\}$, where each $b_i \in X_W^C$.

Due to the fact that the b_i's are B-linearly independent, it follows from proposition (A.1.3) that $b_1(k), \ldots, b_n(k)$ are K-linearly independent for each k in Z. That is,

$$\text{rank } [b_1(k), b_2(k), \ldots, b_n(k)] = n \quad \text{for all } k \text{ in } Z. \tag{6.38}$$

By definition of basis, we have that each elementary column c_i of $\mathbf{B}_W$ can be expressed uniquely as a B-linear combination of $b_1, b_2, \ldots, b_n$. In particular, that means

$$c_i = \sum_{j=1}^{n} a_{ji} b_j \qquad i = 1, 2, \ldots, n$$

for some a_{ji}'s in B.

Therefore, for each $k \in Z$ we can write the following matrix relation

$$[c_1(k), c_2(k), \ldots, c_{\mu m}(k)] = [b_1(k), \ldots, b_n(k)]A(k) \tag{6.39}$$

where $A(k) := (a_{ij}(k))_{\mu m \times \mu m}$.

Next, let J_μ denote the matrix consisting of the first μp-rows of $[b_1, \ldots, b_n]$. Then, it is not hard to see from (6.39) that

$$B_W^{(\mu)}(k) = J_\mu(k)A(k) \text{ for each } k \in Z. \tag{6.40}$$

By applying Sylvester's inequality to (6.40) and by noticing that J_μ is a submatrix of $[b_1, \ldots, b_n]$ it is easy to show that

$$\text{rank } B_W^{(\mu)}(k) = n \text{ for each } k \in Z \quad . \tag{6.41}$$

On the other hand, since χ_W^c is generated by $c_1, c_2, \ldots, c_{\mu m}$ we have that for each $i \in \{1, 2, \ldots, n\}$

$$b_i = \sum_{j=1}^{\mu m} t_{ji} c_i, \quad t_{ji}\text{'s in } B \quad .$$

Hence, at each $k \in Z$ we can write the following matrix relation

(6.42) $[b_1(k), b_2(k), \ldots, b_n(k)] = [c_1(k), c_2(k), \ldots, c_{\mu m}(k)]T(k)$,

where $T(k) := (t_{ij}(k))_{\mu m \times n}$.

Define as $J_i(C_i)$ the $ip \times n$ $(ip \times \mu m)$ matrix formed by the first ip-elementary rows of $[b_1, b_2, \ldots, b_n]$ (of $[c_1, c_2, \ldots, c_{\mu m}]$), and use (6.42) to obtain

$$J_i(k) = C_i(k)T(k)$$

From Sylvester's inequality applied to the above equality, we obtain that for each $i \in Z^+$

$$\text{rank } J_i(k) \leq \text{rank } C_i(k) \text{ for each } k \in Z.$$

Also, note that the following relations are true

$$[b_1, \ldots, b_n] = J_\infty$$

$$[c_1, \ldots, c_{\mu m}] = C_\infty.$$

Hence, from the above inequality we get

(6.43) $n = \text{rank } [b_1(k), \ldots, b_n(k)] \leq \text{rank } [c_1(k), \ldots, c_n(k)]$, $k \in Z$.

But from the relation

$$\text{rank } B_W^{(i)}(k) = \text{rank } B_W^{(\mu)}(k),\ k \in Z \quad ,$$

it follows that rank $B_W^{(\mu)}(k)$ = rank $[c_1(k), \ldots, c_n(k)]$, for each $k \in Z$. And from (6.43) we have

$$n \leq \text{rank } B_W^{(\mu)}(k) \text{ for all } k \in Z \quad .$$

Therefore, we have shown that

$$\text{rank } B_W^{(i)}(k) = n \text{ for all } i \geq \mu, \text{ and all } k \in Z \quad ,$$

i.e., f_W has finite constant rank by definition.

By combining propositions (6.33) and (6.37) we obtain the following criteria to determine when an i/o map f_W has finite constant rank.

(6.44) THEOREM: A linear i/o map $f_W: R_+^m \to R_+^p$ with unit-pulse response matrix $W(\cdot,\cdot)$ has constant rank n if, and only if, the B-modules X_W^r and X_W^c are finitely generated and free with rank n.

Combining the above results we have the following equivalent conditions for realizability

(6.45) THEOREM: Given a linear input/output map f_W: $\mathcal{B}_+^m \to \mathcal{B}_+^p$ with associated Hankel matrix B_W, the following statements are equivalent

(i) f_W has finite constant rank equal to n; i.e., rank $B_W^{(i)}(k) = n$, for all $i > \mu$, all $k \in Z$, for some positive integer μ;

(ii) f_W has a canonical realization of dimension n;

(iii) the $\mathcal{B}$-modules X_W^r and X_W^c are finitely generated and free with rank X_W^r = rank X_W^c = n.

CHAPTER SEVEN
CANONICAL REALIZATIONS

In this section we shall find an abstract realization of an i/o map f_W with finite constant rank which is the discrete-version of one given in KAMEN [1979] for analytic integral operators.

Denote by B^∞ the set of all infinite column vectors over B. By defining vector addition and scalar multiplication componentwise, it is easily shown that B^∞ becomes a B-module. Given a fixed positive integer p, define the following transformation on B^∞

$$\psi_p : B^\infty \to B^\infty : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \dashrightarrow \sigma \begin{bmatrix} x_{1+p} \\ x_{2+p} \\ x_{3+p} \\ \vdots \end{bmatrix}$$

where σ is the right-shift operator on B.

It is very easy to show that ψ_p is additive, and that for any $a \in B$, $x \in B^\infty$

$$\psi_p(ax) = (\sigma a)\psi_p(x)$$

Therefore, we observe that ψ_p becomes a semilinear transformation on $\mathcal{B}^\infty$ with respect to σ. Note ψ_p is the discrete-time counterpart to the pseudo-linear transformation λ_p defined in the theory of time-varying analytic systems (see KAMEN [1979]).

Let f_W: $\mathcal{B}_+^m \to \mathcal{B}_+^p$ be linear i/o map which has finite constant rank equal to n; that is,

(7.2) $\quad$ rank $B_W^{(i)}(k) = n$ for all $i > \mu$, and all $k \in Z$,

for some finite nonnegative integer n.

Consider the B-modules X_W^r and X_W^c associated with $\mathcal{B}_W$ (recall that $X_W^r(X_W^c)$ is the B-module generated by the elementary rows (columns) of B_W). It must be clear that X_W^c is a B-submodule of $\mathcal{B}^\infty$. Hence, it makes sense to study further properties of ψ_p based upon its effects on elements of X_W^c.

Given a matrix $M = (m_1, m_2, \ldots, m_n)$ with $m_i \in \mathcal{B}^\infty$, we shall denote by $\psi_p(M)$ the matrix $(\psi_p(m_1), \psi_p(m_1), \ldots, \psi_p(m_n))$.

(7.3) PROPOSITION: Let c_i denote the i^{th}-elementary column of the generalized Hankel matrix associated with a linear i/o map f_W: $\mathcal{B}_+^m \to \mathcal{B}_+^p$. Then

(7.4) $\quad \psi_p(c_i) = c_{i+m}$ for each $i \in Z^+$

PROOF: Denote by C_j denote the j^{th}-block column of the Hankel matrix B_W; that is $C_j^T = [W_{1j}^T, W_{2j}^T, \ldots]$. Then, it follows immediately from (7.1) that

$$\phi_p(C_j) = \begin{bmatrix} \sigma W_{2j} \\ \sigma W_{3j} \\ \vdots \end{bmatrix} . \tag{7.5}$$

By lemma (4.13), we know that

$$\sigma W_{i+1,j} = W_{i,j+1} \text{ for all } i,j \in Z^+ .$$

Therefore $\phi_p(C_j)$ in (7.5) becomes

$$\phi_p(C_j) = \begin{bmatrix} \cdot \\ W_{1,j+1} \\ W_{2,j+1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \text{for each } j \in Z^+ .$$

From such a relation, it is not hard to see that $\phi_p(c_i) = c_{m+i}$.

□

(7.6) COROLLARY: χ_W^c is a ϕ_p-invariant B-submodule of B^∞. That is,

$$\phi_p(\chi_W^c) \subseteq \chi_W^c .$$

PROOF: Let $x \in X_W^c$ be completely arbitrary. By definition of X_W^c, x can be expressed as

$$x = \sum_{i=1}^{\infty} a_i c_i$$

where the family $(a_i : i \in Z^+)$ is a finite subset of B, i.e., almost all a_i's are zero. Therefore

$$\phi_p(x) = \phi_p\left(\sum_{i=1}^{\infty} a_i c_i\right) = \sum_{i=1}^{\infty} (\sigma a_i) c_{i+m} \qquad \text{(from (7.4))}$$

That is,

$$\phi_p(x) \in X_W^c \quad .$$

□

Since X_W^c is ϕ_p-invariant, ϕ_p can be restricted to X_W^c to obtain a semilinear transformation

$$\bar{\phi}_p : X_W^c \to X_W^c : x \to \phi_p(x) \tag{7.7}$$

By using $\bar{\phi}_p$ we shall be able to construct a minimal realization for a given i/o map f_W satisfying (7.2). In fact, when f_W has finite constant rank n, by theorem (6.44) X_W^c is a finitely generated free module over B with rank n. This means that X_W^c admits a basis $\{b_1, b_2, \ldots, b_n\}$ with each $b_i \in X_W^c$. Therefore, any element x can be expressed uniquely as

$$x = \sum_{i=1}^{n} x_i b_i, \; x_i \in B. \tag{7.8}$$

As a consequence of corollary (7.6), for each $i \in \{1, 2, \ldots, n\}$ we have that $\bar{\Phi}_\rho(b_i) \in X_W^c$. Hence, because of (7.8)

$$\bar{\Phi}_\rho(b_i) = \sum_{j=1}^{n} d_{ij} b_j \text{ for some } d_{ij}\text{'s} \in B.$$

Let D denote the n x n matrix over B defined by

$$D := (d_{ij})_{n \times n} \; . \tag{7.9}$$

Note that D is simply the matrix representation of $\bar{\Phi}_\rho$ with respect to the basis $\{b_1, b_2, \ldots, b_n\}$ of X_W^c.

Next, let B^m denote the B-module consisting of all m-element column vectors over B. It should be clear that B^m is finitely generated and free with rank m. In fact, the unit vectors $e_1, e_2, \ldots, e_m$ in B^m form a basis for such a module.

Let $\bar{E}$ and $\bar{H}$ denote the following B-module homomorphisms

$$\bar{E}: B^m \to X_W^c: \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \to \sum_{i=1}^{m} u_i c_i \quad , \tag{7.10}$$

$$(7.11) \qquad \bar{H}: X_X^c \to B^p: \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} .$$

And denote by E (respectively H) the matrix representation of $\bar{E}(\bar{H})$ with respect to the basis $\{b_1, b_2, \ldots, b_n\}$ in X_W^c and the standard basis $e_1, e_2, \ldots, e_m$ in B^m $\{(e_1, e_2, \ldots, e_p)$ in $B^p\}$.

Our next step is to show that the n-dimensional system $\Sigma = (D,E,H)$ is in fact a realization of f_W. From proposition (4.8) we know that Σ is a realization of f_W if, and only if,

$$(7.12) \qquad W_{ij} = HS^{j-1}E \text{ for all } j \in Z^+ .$$

Now, due to the fact that D, E, H are respectively the matrix representations of $\bar{\psi}_p$, $\bar{E}$, and $\bar{H}$, condition (7.12) is equivalent to

$$(\bar{H}(\bar{\psi}_p^{j-1}(\bar{E})))(e_i) = \text{ith-column of } W_{ij}$$

for each $i \in \{1, 2, \ldots, m\}$ and $j \in Z^+$.

From (7.10), it is seen that

$$(7.13) \qquad \bar{E}(e_i) = \sum_{j=1}^{m} \Delta(i-j)c_j = c_i \text{ for each } i \in \{1, 2, \ldots, m\}$$

where

$$\Delta(i) = \begin{cases} 1, & i=0 \\ 0, & \text{otherwise} \end{cases} .$$

But, by proposition (7.3) we know that $\bar{\Phi}_\rho(c_i) = c_{i+m}$. And, it can be shown by induction on j the following relation

(7.14) $\bar{\Phi}_\rho^{j-1}(c_i) = c_{i+(j-1)m}$, for each $j \in Z^+$, each $i \in \{1,2, \ldots, m\}$.

Now, combine (7.13) and (7.14) to obtain

$$(\bar{\Phi}_\rho^{j-1}\bar{E})(e_i) = c_{i+(j-1)m} .$$

Finally, from the definition of $\bar{H}$ (see (7.11)) we have

$$(\bar{H}\bar{\Phi}_\rho^{j-1}E)(e_i) = \text{first } \rho\text{-elements of } c_{i+(j-1)m} ,$$

which is in fact the i^{th}-column of W_{ij}.

We have thus shown that $\Sigma = (D,E,H)$ is an n-dimensional realization of f_W. Furthermore, from theorem (6.45) is follows that Σ is a minimal canonical realization of such an i/o map.

It should be noted that when using the above procedure to construct a realization for f_W, it does not matter if you use X_W^r or X_W^c. And the reason is that when f_W has finite constant rank, X_W^r and X_W^c have the same rank (see theorem (6.44)). Therefore, in either case you get a (minimal) canonical realization. This is a particular feature of

discrete-time systems which does not exist for analytic time-varying ones since in the latter $X_W^r(X_W^c)$ may be free while $X_W^c(X_W^r)$ is not (see KAMEN [1979] for details).

CHAPTER EIGHT
COMPUTATION OF CANONICAL REALIZATIONS

In this section, we give an algorithm to construct canonical realizations of input/output maps $f_W: B_+^m \to B_+^p$ with finite constant rank equal to n. This result is the discrete version of Kamen's algorithm for constructing minimal realizations of real analytic time-varying integral operators. Therefore, it is not surprising that the major problem here is the computation of a basis for the module X_W^c which by theorem (6.45) is finite and free.

Given a linear response map $f_W: B_+^m \to B_+^p$ with finite constant rank equal to n; that is, there exists a positive integer μ such that

$$\text{rank } B_W^{(i)}(k) = n \text{ for all } i \geq \mu, \text{ and all } k \in Z$$

By theorem (6.45), both modules X_W^r and X_W^c are finitely generated and free with rank n. Therefore, there exist positive integers q and s such that

$$X_W^c = \langle c_1, c_2, \ldots, c_{qm} \rangle_B$$

$$X_W^r = \langle r_1, r_2, \ldots, r_{\rho s} \rangle_B$$

where $c_i(r_i)$ is the i^{th}-elementary column (row) of $\mathcal{B}_W$ (the generalized Hankel matrix associated with f_W).

Let $\mu := \max(q,s)$, and let Γ be any $s \times q$ submatrix of $B_W^{(\mu)}$ such that

(i) $s > p$,

(ii) $$B_W^{(\mu)} = \left[\begin{array}{c:c:c} \cdot & \Gamma & \cdot \\ \cdot & \cdot & \cdot \\ \hdashline \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right] ,$$

with rank $\Gamma(k) = n$ for all $k \in Z$.

By using the discrete-time version of Dolezal's theorem (see theorem (A.1.14) in the appendix), there exist matrices $\Lambda \in B^{q \times q}$ and $Q \in B^{s \times n}$ such that

(i) $\Gamma\Lambda = [Q;\ 0_{s \times (q-n)}]$,

(ii) Λ is invertible over B,

(iii) rank $Q(k) = n$ for all $k \in Z$.

It follows from corollary (A.1.16) that there exists a basis $\{b_1, b_2, \ldots, b_n\}$ for X_W^c satisfying

$$[b_1, b_2, \ldots, b_n] = \left[\begin{array}{c} Q \\ \hdashline \\ \\ \end{array}\right] . \tag{8.1}$$

Next, consider the mappings $\tilde{\Phi}_\rho$, $\tilde{E}$, $\tilde{H}$ defined by (7.7), (7.10) and (7.11) respectively. From the definition of $\tilde{E}$, it must be clear that

(8.2) $[\tilde{E}(e_1),\tilde{E}(e_2), \ldots, \tilde{E}(e_n)] = C_1 = [b_1,b_2, \ldots, b_n]E$ (see (7.13))

where C_1 denote the first block-column of the Hankel matrix B_W.

Let Ω denote the submatrix of the first block column of $B_W^{(\mu)}$ corresponding to the rows of Γ. Then from (8.1), it is clear that

(8.3) $\Omega = QE$.

Premultiply both sides of (8.3) by the left-inverse of Q to obtain

(8.4) $E = (Q^TQ)^{-1}Q^T\Omega$.

Also, from the definition of $\tilde{H}$ (see (7.11))

$[\tilde{H}(b_1),\tilde{H}(b_2), \ldots, \tilde{H}(b_n)]$ = first ρ-rows of $[b_1,b_2, \ldots, b_n] = H$.

By using (8.1), we observe that

(8.5) H = first ρ rows of Q

On the other hand, from the definition of $\bar{\phi}_p$ (see also (7.9))

$$[\bar{\phi}_p(b_1),\bar{\phi}_p(b_2), \ldots, \bar{\phi}_p(b_n)] = [b_1,b_2, \ldots, b_n]D$$

because of (8.1), we can restrict $\bar{\phi}_p$ to the columns of Q to get

(8.6) $$\bar{\phi}_p(Q) = QD \quad .$$

Next recall that

(i) $\phi_p(ax) = (\sigma a)\phi_p(x)$ for each $a \in B$, and each $x \in \chi_W^c$.

(ii) $\Gamma\Lambda = [Q;\ 0_{sx(q-n)}]$.

Then, from (8.6) it is not hard to see that

(8.7) $$[\bar{\phi}_p(Q);\ 0_{sx(q-n)}] = \bar{\phi}_p(\Gamma\Lambda) = \Psi_p(\Gamma)(\sigma\Lambda) \quad ,$$

where

σ = right-shift operator,

$\Psi_p(\Gamma)$ = submatrix of $B_W^{(\mu+1)}$ whose first column is located m-elementary columns to the right of the first column of Γ, and whose rows correspond to those of Γ.

Finally, combine (8.6) and (8.7) to obtain

(8.8) D = first n-columns of $(Q^T Q)^{-1} Q^T \Psi_p(\Gamma)(\sigma\Lambda)$.

Now, from results of Chapter 7 the system $\Sigma = (D,E,H)$ given by equations (8.8), (8.4) and (8.5) is a minimal canonical realization of f_W.

It is worthwhile to make the following comments about the above realization algorithm:

(i) The most difficult step (from the standpoint of computation) is the determination of the matrices Λ and Q, or equivalently the construction of a basis for the B-module X_W^C. In general, it is not possible to find those matrices in a finite number of steps because of the structure of the ring B (at each $k \in Z$, $\Lambda(k)$ and $Q(k)$ can be found by reducing $\Gamma(k)$ to its column echelon form). However, it follows from known results of linear algebra, that when the entries of the unit pulse response $W(\cdot,\cdot)$ belong to a subring A of B, A being a Bezout domain, Λ and Q can be determined via a finite sequence of multiplication of Γ using $q \times q$ invertible matrices over A. For example, when $K = R$ (the field of real numbers) and the subring A may be taken as $R[k]$, the ring of polynomials with real coefficients, or as the subring consisting of all finite sums of sequences of the form $e^{a_i k} \cos b_i k \sin c_i k$ with a_i, b_i, $c_i \in R$.

(ii) Weiss's algorithm in theorem (2.3.44) can be viewed as a particular case of the realization procedure presented here. For the

former, it is needed that $B_W^{(\mu)}$ contain an n x n submatrix with rank n at each k ε Z, while for the latter the submatrix Γ need not be n x n. Below, we give an example to illustrate not only our realization procedure, but also to show that the assumption that there is an n x n invertible submatrix of $B_W^{(\mu)}$ may not be true; even though f_W has finite constant rank.

(iii) If f_W has a time-invariant realization, then the algorithm will give a constant minimal realization.

(8.10) EXAMPLE: Consider the scalar input/output map f_W: $\mathcal{B}_+ \to \mathcal{B}_+$ with unit-pulse response $W(k,l) = \sin \frac{\pi}{2} k \cos \frac{\pi}{2} (l+1)$ for all $(k,l) \in CP$. The Hankel matrix B_W associated with f_W is given by

$$B_W(k) = \begin{bmatrix} 0 & \sin^2 \frac{\pi}{2} k & 0 & -\sin^2 \frac{\pi}{2} k & \cdots \\ \cos^2 \frac{\pi}{2} k & 0 & -\cos^2 \frac{\pi}{2} k & 0 & \\ 0 & -\sin^2 \frac{\pi}{2} k & 0 & \sin^2 \frac{\pi}{2} k & \cdots \\ -\cos^2 \frac{\pi}{2} k & 0 & \cos^2 \frac{\pi}{2} k & 0 & \\ 0 & \sin^2 \frac{\pi}{2} k & 0 & -\sin^2 \frac{\pi}{2} k & \cdots \\ \cos^2 \frac{\pi}{2} k & 0 & -\cos^2 \frac{\pi}{2} k & 0 & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad k \in Z.$$

It is easily seen that

$$\text{rank } \mathcal{B}_W^{(i)}(k) = 1 \text{ for all } i \geq 2, \text{ and all } k \in Z.$$

That is, f_W has finite-constant rank equal to one.

Note that Weiss algorithm can not be used here to find a canonical realization of f_W because there is not a 1x1 submatrix of $B_W^{(2)}$ with rank one at each $k \in Z$. However, we can still use our procedure. In fact take $\Gamma(k) = B_W^{(2)}(k)$. By performing a sequence of elementary column operations on Γ we can find that

$$\Delta(k) = \begin{bmatrix} \cos^2 \frac{\pi}{2} k & \sin^2 \frac{\pi}{2} k \\ \sin^2 \frac{\pi}{2} k & \cos^2 \frac{\pi}{2} k \end{bmatrix}$$

and

$$Q(k) = \begin{bmatrix} \sin^2 \frac{\pi}{2} k \\ \cos^2 \frac{\pi}{2} k \end{bmatrix} .$$

The left-inverse of Q is equal to

$$(Q^T(k)Q(k))^{-1}Q^T(k) = [\sin^2 \tfrac{\pi}{2} k \ \cos^2 \tfrac{\pi}{2} k] \quad .$$

Also, we have that

$$\Omega(k) = \begin{bmatrix} 0 \\ \cos^2 \frac{\pi}{2} k \end{bmatrix} ,$$

$$\psi_p(\Gamma)(k) = \begin{bmatrix} \sin^2 \frac{\pi}{2} k & 0 \\ 0 & -\cos^2 \frac{\pi}{2} k \end{bmatrix},$$

$$(\sigma\Lambda)(k) = \begin{bmatrix} \sin^2 \frac{\pi}{2} k & \cos^2 \frac{\pi}{2} k \\ \cos^2 \frac{\pi}{2} k & \sin^2 \frac{\pi}{2} k \end{bmatrix}.$$

Finally, from formulas (8.4), (8.8) and (8.5) we find the minimal canonical realization $\Sigma = (d,e,h)$

$$d(k) = \sin^2 \frac{\pi}{2} k - \cos^2 \frac{\pi}{2} k \quad ;$$

$$e(k) = \cos^2 \frac{\pi}{2} k \quad ;$$

$$h(k) = \sin^2 \frac{\pi}{2} k \quad .$$

We shall conclude this section with an application of the algorithm developed here to system reduction. A discrete time-varying system $\Sigma = (D,E,H)$ over B is said to be reducible if Σ is not a minimal realization of its response map f_Σ defined by (2.3.7). However, in order to reduce a given Σ by using our realization procedure it is required that f_Σ has finite constant rank.

Given an $\bar{n}$-dimensional system $\Sigma = (D,E,H)$ over B, let B_W denote the (generalized) Hankel matrix associated with the system's input/output

map f_Σ (see (2.3.14)), and let R_i, O_i denote the i^{th}-reachability and i^{th}-observability matrix of Σ; that is

$$R_i = [E, SE, \ldots, S^{i-1}E] \quad ,$$

$$O_i = [H^T, \hat{S}H^T, \ldots, \hat{S}^{i-1}H^T]^T \quad .$$

(8.11) PROPOSITION: Let $\Sigma = (D,E,H)$ be a finite-dimensional system over B for which there exists positive integers α, β such that

(8.12) $\quad \text{rank } R_i(k) = \text{rank } R_\alpha(k)$ for all $i \geq \alpha$, and all $k \in Z$,

(8.13) $\quad \text{rank } O_i(k) = \text{rank } O_\beta(k)$ for all $i \geq \beta$, and all $k \in Z$.

Then

(8.14) $\quad \text{rank } B_W^{(i)}(k) = \text{rank } B_W^{(\mu)}(k)$ for all $i \geq \mu$, and all $k \in Z$,

where $\mu := \max\{\alpha,\beta\}$, and $B_W^{(i)}(k)$ is the submatrix of B_W given by

$$B_W^{(i)}(k) = \begin{bmatrix} W_{11}(k) & W_{12}(k) & \ldots & W_{1i}(k) \\ W_{21}(k) & W_{22}(k) & \ldots & W_{2i}(k) \\ \vdots & & & \\ W_{i1}(k) & W_{i2}(k) & \ldots & W_{ii}(k) \end{bmatrix} \quad ; \quad k \in Z \quad .$$

and

$$W_{ij}(k) = W(k+i-1,\ k-j) \quad \text{for all } k \in Z, \text{ and all } i,j \in Z^+ \quad .$$

PROOF:

Since Σ is a realization of its i/o map f_Σ, it follows from proposition (6.9) that

$$B_W^{(i)} = O_i R_i \quad \text{for each } i \in Z^+ \quad . \tag{8.15}$$

Let $\mu := \max(\alpha,\beta)$, then the relation (8.12) says, in particular, that

$$\text{rank } R_{\mu+1}(k) = \text{rank } R_\mu(k) \quad \text{for all } k \in Z \quad .$$

Hence, there exists a $\mu m \times m$ matrix Λ over $\mathcal{B}$ such that

$$S^\mu E = R_\mu \Lambda \quad . \tag{8.16}$$

Likewise, it can be shown that relation (8.13) implies that there is a $p\times\mu p$ matrix Γ over $\mathcal{B}$ such that

$$(\hat{S}^\mu H^T)^T = \Gamma O_\mu \quad . \tag{8.16}$$

From (8.14) we observe that $B_W^{(\mu+1)}$ can be written as

$$B_W^{(\mu+1)} = O_{\mu+1} R_{\mu+1}$$

$$= \begin{bmatrix} 0_\mu \\ (\hat{S}^\mu H^T)^T \end{bmatrix} \begin{bmatrix} R_\mu & S^\mu E \end{bmatrix} \quad \text{(By definiton of } R_i \text{ and } 0_i\text{)}$$

$$= \begin{bmatrix} 0_\mu R_\mu & 0_\mu (S^\mu E) \\ (\hat{S}^\mu H^T)^T R_\mu & (\hat{S}^\mu H^T)^T (S^\mu E) \end{bmatrix}$$

$$= \begin{bmatrix} B_W^{(\mu)} & 0_\mu (S^\mu E) \\ (\hat{S}^\mu H^T)^T R_\mu & (\hat{S}^\mu H^T)^T (S^\mu E) \end{bmatrix} \quad \text{(By (8.14))} .$$

By using (8.15-16) and (8.14) we have then that

$$B_W^{(\mu+1)} = \begin{bmatrix} B_W^{(\mu)} & B_W^{(\mu)} \Lambda \\ \Gamma B_W^{(\mu)} & \Gamma B_W^{(\mu)} \Lambda \end{bmatrix} . \tag{8.17}$$

Now, due to the fact that $B_W^{(\mu)}$ is a submatrix of $B_W^{(\mu+1)}$ (see (8.17)), it is clear that

$$\text{rank } B_W^{(\mu+1)}(k) \geq \text{rank } B_W^{(\mu)}(k) \quad \text{at each } k \in Z .$$

On the other hand, observe that the right-hand side of (8.17) an be written as $\begin{bmatrix} I \\ \Gamma \end{bmatrix} B_W^{(\mu)} [I \ \Lambda]$, where I = identity matrix. Therefore,

$$(8.18) \qquad B_W^{(\mu+1)} = \begin{bmatrix} I \\ \Gamma \end{bmatrix} B_W^{(\mu)} \; [I \; \Lambda] \quad .$$

By using Sylvester's inequality we obtain from (8.18) that

$$\text{rank } B_W^{(\mu+1)}(k) \leq \text{rank } B_W^{(\mu)}(k) \quad \text{at each } k \in Z \quad .$$

We have thus proved that

$$\text{rank } B_W^{(\mu+1)}(k) = \text{rank } B_W^{(\mu)}(k) \text{ at each } k \in Z \quad .$$

As a result, the relation (8.14) follows immediately by using induction on i.

□

As examples of systems satisfying the hypothesis of proposition (8.11) we have those systems over B that have constant rank.

(8.19) THEOREM: Let $\Sigma = (D,E,H)$ be an $\bar{n}$-dimensional system over B satisfying the hypothesis of proposition (8.11). Then, the i/o map f_Σ associated with Σ has a canonical realization if and only if there exists a positive integer $n \leq \bar{n}$ such that

$$(8.20) \qquad \text{rank } B_W^{(\mu)}(k) = n \text{ for all } k \in Z \quad .$$

Furthermore, if (8.20) holds a canonical realization of f_Σ can be computed from $B_W^{(\mu+1)}$.

PROOF: Suppose f_Σ has a canonical realization of dimension n, by proposition (6.24) f_Σ must have finite-constant rank. Therefore, it follows from definition (6.9) and proposition (6.11) that

$$\text{rank } B_W^{(\mu)}(k) = n \text{ for all } k \in Z \quad .$$

Conversely, assume that relation (8.20) holds. Since f_Σ satisfies (8.14), f_Σ has finite costant rank equal to n (by definition). Therefore, by theorem (6.32), f_Σ is realizable by a canonical system.

Finally, the last statement of the theorem is proved by noticing that when f_Σ has finite-constant rank (that is, (8.20) holds), a minimal (canonical) realization can be computed by using the realization procedure developed in this section, and in order to do that, we need only to know $B_W^{(\mu+1)}$.

□

(8.21) EXAMPLE: Consider the system $\Sigma = (D,e,h)$ over $\mathcal{B}$ given by

$$D(k) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{(|k|+1)(|k-1|+1)} & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e(k) = \begin{bmatrix} 1 \\ 0 \\ \sin \frac{\pi}{2} k \end{bmatrix} \quad ; \quad h(k) = \begin{bmatrix} 1 \\ |k| + 1 \\ 1 \end{bmatrix} .$$

It is not hard to show that Σ is a constant rank system with $\alpha = 2$, $\beta = 1$. Furthermore, rank $B_W^{(2)}(k) = 1$ for all $k \in Z$ (here $\mu = 2$), where

$$B_W^{(2)}(k) = \begin{bmatrix} 1 + \sin \frac{\pi}{2} k & 1 - \cos \frac{\pi}{2} k \\ 1 + \sin \frac{\pi}{2} K & 1 - \cos \frac{\pi}{2} k \end{bmatrix}.$$

Then, by theorem (8.19), f_Σ has a canonical realization whose dimension is one (see theorem (6.45)), and in order to compute one we need only to know

$$(8.22) \qquad B_W^{(3)}(k) = \begin{bmatrix} 1 + \sin \frac{\pi}{2} k & 1 - \cos \frac{\pi}{2} k & 1 - \sin \frac{\pi}{2} k \\ 1 + \sin \frac{\pi}{2} k & 1 - \cos \frac{\pi}{2} k & 1 - \sin \frac{\pi}{2} k \\ 1 + \sin \frac{\pi}{2} k & 1 - \cos \frac{\pi}{2} k & 1 - \sin \frac{\pi}{2} k \end{bmatrix}.$$

It is interesting to observe from (8.22) that Weiss's algorithm can not be used in this case because there is not a fixed 1x1 invertible submatrix of $B_W^{(3)}$. However, our realization procedure can be used. In fact, take $\Gamma(k) = B_W^{(2)}(k)$ since rank $B_W^{(2)}(k) = 1$ for all $k \in Z$. Applying the basis construction, we obtain that

$$\Lambda(k) = \begin{bmatrix} 1 & 1 - \cos \frac{\pi}{2} k \\ 1 & -(1 + \sin \frac{\pi}{2} k) \end{bmatrix}; \quad Q(k) = \begin{bmatrix} 2 + \sin \frac{\pi}{2} k - \cos \frac{\pi}{2} k \\ 2 + \sin \frac{\pi}{2} k - \cos \frac{\pi}{2} k \end{bmatrix}.$$

Hence,

$$\Omega(k) = \begin{bmatrix} 1 + \sin \frac{\pi}{2} k \\ \\ 1 + \sin \frac{\pi}{2} k \end{bmatrix},$$

$$[\Phi_1(\Gamma)](k) = \begin{bmatrix} 1 - \cos \frac{\pi}{2} k & 1 - \sin \frac{\pi}{2} k \\ \\ 1 - \cos \frac{\pi}{2} k & 1 - \sin \frac{\pi}{2} k \end{bmatrix},$$

$$(\sigma\Lambda)(k) = \begin{bmatrix} 1 & 1 - \sin \frac{\pi}{2} k \\ \\ 1 & -1 + \cos \frac{\pi}{2} k \\ \cdot & \end{bmatrix}.$$

Finally, by evaluating formulas (8.4), (8.8) and (8.5) the following canonical (minimal) realization of f_Σ is obtained.

$$d(k) = \frac{2 - (\cos \frac{\pi}{2} k + \sin \frac{\pi}{2} k)}{2 + \sin \frac{\pi}{2} k - \cos \frac{\pi}{2} k},$$

$$e(k) = \frac{1 + \sin \frac{\pi}{2} k}{2 + \sin \frac{\pi}{2} k - \cos \frac{\pi}{2} k},$$

$$h(k) = 2 + \sin \frac{\pi}{2} k - \cos \frac{\pi}{2} k .$$

CHAPTER NINE
CONCLUSIONS AND OPEN PROBLEMS

By introducing the concept of generalized representative series into the theory of discrete time-varying systems we have been able to find necessary and sufficient conditions for realizability given in terms of the Hankel matrix. Such criteria can be considered "local" because they require the computation of the rank of an infinite matrix at each instant of time k.

In addition to the local realizability criteria, the algebraic framework developed in this work yields global criteria for a linear discrete time-varying i/o map to be realizable.

This global approach to realizability provides additional insight into the structural characteristics of the problem in the discrete time-varying case. In fact, the results of Chapters Two and Three show how realizability criteria in the time-invariant case, and in the real analytic time-varying case, generalize to criteria in the discrete time-varying framework.

We have also answered the important question as to which kinds of i/o maps always have a canonical realization, and we have shown that for only those maps a realization is canonical if and only if it is

minimal. Furthermore, it has been seen that the construction of such state-space models reduces to the problem of computing a basis for a finitely-generated free module over the ring B.

As it was noticed, there is a lot of similarity between the algebraic approach developed here and the module approach to realizability in the theory of real analytic time-varying systems. Actually, the reason for such a similarity rests on the fact that both theories are given in terms of pseudo-linear transformations which are not linear with respect to the module structure of B and A(J) respectively. It was this observation made by KAMEN [1979] that motivated most of the work done in this dissertation.

There are several aspects of the approach taken here which deserve investigation:

(P.1) Let $\Sigma = (D,E,H)$ be an n-dimensional system over B, and let f_Σ be its i/o map. Suppose that Σ is a constant rank system with "parameters" α, β, n_r, n_o (see definition (6.1)) with $D \neq 0$. Does f_Σ have finite-constant rank? We have shown that when $D = 0$, the answer to that question is generally "no". However, most of the examples with $D \neq 0$ tell us that with $\mu = \max(\alpha, \beta)$

$$\text{rank } B_W^{(i)}(k) = \min(n_r, n_o) \quad \text{for all } i \geq \mu, \text{ and all } k \in Z \quad .$$

(P.2) Given a realizable i/o map f_W which does not have constant rank, develop an algorithm to construct a minimal realization Σ.

(P.3) Can the canonical realization procedure given here be specialized so that it yields a stable realization with bounded coefficients whenever the given i/o map is bounded-input/bounded-output (BIBO) stable.

(P.4) How can we construct "partial" realizations of any i/o map? The greatest difficulty with this topic consists in giving an appropriate definition of the problem in such a way that it has a feasible "physical" interpretation, and that it generalizes the existing one for time-invariant systems. Another important step to the solution of this problem will be to choose the "right" algebraic methodology. Unfortunately, the algebraic framework developed in this work does not look promising to obtain the desired answers; however, it is believed that a good technique would be the theory of non-commutative formal power series (see FLIESS [1974],[1976],[1978]).

APPENDIX ONE

DISCRETE VERSION OF DOLEZAL'S THEOREM

Let B denote the ring of all K-valued bi-infinite sequences. Also, let M be any B-module. Given a finite set of elements in M, say $\xi = \{m_1, m_2, \ldots, m_n\}$ we have the following

(A.1.1) DEFINITION: A finite set ξ is said to be B-linearly dependent if there exists scalars $r_1, r_2, \ldots, r_n$ in B (not all zero) such that

(A.1.2) $$r_1m_1 + r_2m_2 + \ldots + r_nm_n = 0$$

Otherwise, the set ξ is said to be B-linearly independent.

For a fixed postive integer p, B^p will denote the B-module consisting of all p-element column vectors whose entries are elements of B. Furthermore, by B^∞ we will represent the R-module of all infinite column vectors over B.

EXAMPLES: (a) let M = B. It is interesting to observe that the only R-linearly independent sets of B are those formed by single non-zero elements of R. For suppose, you have a set with more than two

elements, say $\{m_1, m_2, \ldots, m_n\} =: \xi$ and $0 \notin \xi$. Then by taking $r_1 = m_2$, $r_2 = -m_1$, $r_3 = r_4 = \ldots = r_n = 0$, we have that (A.1.2) is satisfied.

(b) Here let $M = B^2$. And consider the following vectors in B^2:

$$x_1(k) = \begin{bmatrix} \sin \frac{\pi}{2} k \\ -\cos \frac{\pi}{2} k \end{bmatrix} ; \quad x_2(k) = \begin{bmatrix} \cos \frac{\pi}{2} k \\ \sin \frac{\pi}{2} k \end{bmatrix} , \quad k \in Z \quad .$$

Let $r_1, r_2 \in R$ such that

$$r_1 x_1 + r_2 x_2 = 0 \quad ,$$

i.e.,

$$\begin{bmatrix} \sin \frac{\pi}{2} k & \cos \frac{\pi}{2} k \\ -\cos \frac{\pi}{2} k & \sin \frac{\pi}{2} k \end{bmatrix} \begin{bmatrix} r_1(k) \\ r_2(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \quad k \in Z \quad .$$

But $\det \begin{bmatrix} \sin \frac{\pi}{2} k & \cos \frac{\pi}{2} k \\ -\cos \frac{\pi}{2} k & \sin \frac{\pi}{2} k \end{bmatrix} = \sin^2 \frac{\pi}{2} k + \cos^2 \frac{\pi}{2} k = 1$ for all $k \in Z$.

By using well known results in linear algebra we conclude

$$\begin{bmatrix} r_1(k) \\ r_2(k) \end{bmatrix} = 0 \quad \underline{\text{for all}} \ k \in Z \quad .$$

That is $r_1 = r_2 = 0$.

Therefore, x_1 and x_2 are linearly independent over $\mathcal{B}$.

(c) In order to generalize example (b) to higher dimensions we propose the following not-hard-to-prove result: "Given p vectors x_1, x_2, ..., x_p in $\mathcal{B}^p$. They are $\mathcal{B}$-linear independent if and only if the matrix $X = [x_1; x_2; \ldots; x_p] \in \mathcal{B}^{p \times p}$ is invertible. That is, they are R-linearly independent if $\det X(k) \neq 0$ for all $k \in Z$."

(A.1.3) PROPOSITION: Let $\{m^1, m^2, \ldots, m^n\}$ be an $\mathcal{B}$-linearly independent set of vectors in $\mathcal{B}^p(\mathcal{B}^\infty)$. Then for each $k \in Z$, the set of vectors $\{m^1(k), m^2(k), \ldots, m^n(k)\}$ in $K^p(K^\infty)$ is K-linearly independent.

PROOF: Assume there exists a k_1 in Z such that $m^1(k_1), m^2(k_1), \ldots, m^n(k_1)$ are K-linearly dependent. By definition, there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in K, not all zero, such that

$$\alpha_1 m^1(k_1) + \alpha_2 m^2(k_1) + \ldots + \alpha_r m^n(k_1) = 0 \quad .$$

Define the following functions $r_i : Z \to K$ by

$$r_i(k) = \begin{cases} \alpha_i, & k = k_1, \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots n \quad .$$

Obviously, the r_i's are in B, and not all of them are zero. Furthermore,

$$r_1 m^1 + r_2 m^2 + \ldots + r_n m^n = 0 \quad .$$

Contradiction!

□

(A.1.4) PROPOSITION: Given a set of vectors in $B^p(B^\infty)$, $\{m^1, m^2, \ldots, m^n(k)$ such that for each $k \in Z$, the vectors $m^1(k), m^2(k), \ldots, m^n(k)$ are K-linearly independent. Then the set $\{m^1, m^2, \ldots, m^n\}$ is B-linearly independent.

PROOF:

Let $a_1, a_2, \ldots, a_n$ be any set of scalars in B satisfying the relation

$$a_1 m^1 + a_2 m^2 + \ldots + a_n m^n = 0 \quad .$$

which is equivalent to

(A.1.5) $a_1(k)m^1(k) + a_2(k)m^2(k) + \ldots + a_n(k)m^n(k) = 0$, for each $k \in Z$.

By hypothesis, for each $k \in Z$ the set $m^1(k), \ldots, m^n(k)$ is k-linearly independent. Hence, from (A.1.5) we have

$$a_1(k) = a_2(k) = \ldots = a_n(k) = 0 \quad \text{for each } k \in Z \quad .$$

That is, $a_1 = a_2 = \ldots = a_n = 0$.

□

By combining propositions (A.1.3) and (A.1.4) we get the following.

(A.1.6) LEMMA: A set of vectors $\{m^1, m^2, \ldots, m^n\}$ in $B^p(B^\infty)$ is B-linearly independent if, and only if, for each $k \in Z$ the vectors $m^1(k), m^2(k), \ldots, m^n(k)$ are K-linearly independent.

Also in the sequel we shall need the next result.

(A.1.7) THEOREM: Let B be a nontrivial commutative ring and M be a free, finitely generated B-module. Then there exists a positive integer n such than any basis B of M has cardinality n. Such a number n is called the rank of M.

PROOF:

See KNIGHT [1971].

□

Let Γ be any $p \times q$ matrix over B. We shall denote by $\mathrm{Im}\Gamma$ the B-module generated by the q-columns of Γ. In other words, if

$\Gamma = [\gamma^1, \gamma^2, \ldots, \gamma^q]$ where $\gamma^i = [\gamma_{1i}, \gamma_{2i}, \ldots, \gamma_{pi}]^T \in \mathcal{B}^p$, $i = 1, 2, \ldots, q$; then

$$(A.1.8) \qquad I_m(\Gamma) := \langle \gamma^1, \gamma^2, \ldots, \gamma^q \rangle_G := \{ \sum_{i=1}^{q} a_i \gamma^i : a_i \in R \} .$$

Now we are ready to show the discrete-time version of Dolezal's theorem (which is probably known).

(A.1.9) THEOREM: Let Γ be any pxq matrix over $\mathcal{B}$. The $\mathcal{B}$-module $\mathrm{Im}\Gamma$ is free with rank n if, and only if

$$\text{rank } \Gamma(k) = n < \infty \quad \text{for all } k \in Z .$$

PROOF: (Necessity) Assume that $\mathrm{Im}\Gamma$ is free. By theorem (A.1.7), it has finite rank n. That is, $\mathrm{Im}\Gamma$ admits a basis $b_1, b_2, \ldots, b_n$, where each $b_i \in \mathrm{Im}\Gamma$. Because of (A.1.8) for each $i \in \{1, 2, \ldots, n\}$

$$b_i = \sum_{j=1}^{q} t_{ji} \gamma^j \quad \text{for some } t_{ji}\text{'s in } \mathcal{B} .$$

Let $A := [b_1, b_2, \ldots b_n] \in \mathcal{B}^{p \times n}$,

$$T := [t_{ij}]_{q \times n} .$$

Then, it is not hard to see that

(A.1.10) $A = \Gamma T$.

On the other hand, from the definition of a basis we have that the n column vectors of A are B-linear independent. And by using lemma (A.1.6) together with well known results of linear algebra we conclude

(A.1.11) $\text{rank } A(k) = n \quad \text{for all } k \in Z$.

From relations (A.1.10) and (A.1.11) together with Sylvester's inequality we can show

(A.1.12) $\text{rank } \Gamma(k) \geq n \quad \text{for all } k \in Z$.

Since $\{b_1, b_2, \ldots, b_n\}$ is a basis of $\text{Im}\Gamma$, each γ^i can be expressed uniquely as

$$\gamma^i = \sum_{j=1}^{n} U_{ji} b_j \qquad i = 1, 2, \ldots, q \quad .$$

That is,

(A.1.13) $\Gamma = AU$,

where

$$U := [U_{ij}]_{nxq} \quad .$$

By using Sylvester's inequality again together with (A.1.11) we obtain from (A.1.13)

$$\text{rank } \Gamma(k) \leq n \quad \text{for all } k \in Z \quad .$$

Therefore, we have shown that

$$\text{rank } \Gamma(k) = n \quad \text{for all } k \in Z \quad .$$

(Sufficiency) Here, suppose that rank $\Gamma(k) = n < \infty$ for all $k \in Z$. By definition of rank of a matrix over a field, we have that for each $k \in Z$ there exist columns $\gamma_{i_1}(k)$, $\gamma_{i_2}(k)$, $\cdots$, $\gamma_{i_n}(k)$, of Γ which are K-linearly independent ($\gamma_{i_j}(k)$ means that the position of the column may vary with time).

Next, let us define elements b_j in B^p by

$$b_j(k) = \gamma_{i_j}(k) \quad \text{for each } k \in Z, \quad j \in \{1, 2, \ldots, n\} \quad .$$

Clearly, for each $k \in Z$ the vectors $b_1(k)$, $b_2(k)$, $\ldots$, $b_n(k)$ are K-linearly independent. Hence, from lemma (A.1.6) they are also B-linearly independent.

Observe that for each $k \in Z$ by using permutation of columns of Γ and by deleting some of them we can write

$$[b_1(k); b_2(k); \ldots; b_n(k)] = \Gamma(k)\Delta(k) \quad .$$

That is

$$[b_1;\ b_2;\ \ldots;\ b_n] = \Gamma\Lambda \quad \text{with } \Lambda \in B^{q\times n} \quad .$$

The above relation says that

$$b_i \in I_m(\Gamma) \quad , \qquad i = 1,\ 2,\ \ldots,\ n \quad .$$

In resume: $\{b_1,\ b_2,\ \ldots,\ b_n\}$ satisfies

(i) They are B-linearly independent.

(ii) Each b_i is an element of $\mathrm{Im}\Gamma$.

To show that they form a basis for the B-module $\mathrm{Im}\Gamma$ we need to prove only that

$$\mathrm{Im}\Gamma = \langle b_1,\ b_2,\ \ldots,\ b_n\rangle_B$$

So let $x \in \mathrm{Im}\Gamma$ be completely arbitrary. Hence, there exist scalars α_1, α_2, $\ldots$, α_q in B such that

$$x = \sum_{j=1}^{q} \alpha_j \gamma^i \quad ;$$

that is,

$$x(k) = \sum_{j=1}^{q} \alpha_j(k)\gamma^j(k) \quad , \qquad k \in Z \quad .$$

From the definition of the vectors $b_1, b_2, \ldots, b_n$ we have that

$$\gamma^j(k) = \sum_{\nu=1}^{n} a_{\nu_j}(k)b_\nu(k), \quad a_{\nu j}\text{'s} \in k, \; k \in Z \text{ and } j = 1, 2, \ldots, q \quad .$$

Hence

$$x(k) = \sum_{j=1}^{q} \alpha_j(k) \sum_{\nu=1}^{n} a_{\nu_j}(k)b_\nu(k) \quad , \quad k \in Z \quad ,$$

$$= \sum_{\nu=1}^{n} \left(\sum_{j=1}^{q} \alpha_j(k)a_{\nu_j}(k)\right)b_\nu(k) \quad ,$$

so we define

$$s_\nu(k) = \sum_{j=1}^{q} \alpha_j(k)a_{\nu_j}(k) \quad \text{for each } k \in Z \quad .$$

It must be clear that for each $\nu = 1, 2, \ldots, n$, $s_\nu \in B$. Therefore,

$$x = \sum_{\nu=1}^{n} s_\nu b_\nu \quad ,$$

i.e.,

$$x \in \langle b_1, b_2, \ldots, b_n \rangle_B \quad .$$

But x was completely arbitrary, so we must conclude

$$\mathrm{Im}\Gamma = \langle b_1, b_2, \ldots, b_n \rangle_B \quad .$$

We have thus shown that $\{b_1, b_2, \ldots, b_n\}$ is a basis for the B-module $\mathrm{Im}\Gamma$. That is, $\mathrm{Im}\Gamma$ is a free B-module with rank N.

EXAMPLE: d) Let $\Gamma \in B^{2x3}$ be defined by

$$\Gamma(k) = \begin{bmatrix} \sin \frac{\pi}{2} k & \Delta(k) & -\cos \frac{\pi}{2} k \\ \cos \frac{\pi}{2} k & 0 & \sin \frac{\pi}{2} k \end{bmatrix} , \quad k \in Z \quad .$$

From example (b) we see that rank $\Gamma(k) = 2$ for all $k \in Z$. Also, we know that $\gamma_1(k) = (\sin \frac{\pi}{2} k; \cos \frac{\pi}{2} k)^T$ and $\gamma_3(k) = (-\cos \frac{\pi}{2} k; \sin \frac{\pi}{2} k)^T$ are B-linearly independent. Hence, $\mathrm{Im}\Gamma$ is free with rank two. In fact γ_1, γ_3 form a basis for it.

(A.1.14) THEOREM: Let Γ be any pxq matrix over B such that

(A.1.15) rank $\Gamma(k) = n < \infty$ for all $k \in Z$.

Then there exists a qxq matrix Λ over B whose inverse is also over B with the following property

(A.1.16) $\Gamma\Lambda = [Q;\ 0_{px(q-n)}]$,

where Q is a pxn matrix over B satisfying rank Q(k) = n for all $k \in Z$.

PROOF:

Let k be a fixed integer. $\Gamma(k)$ can be reduced into its column echelon form by a series of elementary column operations. That means we can find a non-singular matrix Λ_k such that

$$\Gamma(k)\Lambda_k = [Q_k \vdots 0_{px(q-n)}] \quad .$$

Furthermore, because of (A.1.15) Q_k is a pxn matrix with rank n. The reason why we use $\Lambda_k(Q_k)$ is to show the fact that for each k such a matrix may vary.

Now, let us define a qxq matrix, Λ, and a qxn matrix Q over B by

$$\Lambda(k) := \Lambda_k \quad ,$$

$$Q(k) := Q_k \quad ,$$

for each $k \in Z$.

It must be clear that Λ and Q satisfy the conditions of the theorem. That is

(i) $\Gamma\Lambda = [Q;\ 0_{qx(p-n)}]$,

(ii) Γ^{-1} exists over B,

(iii) rank Q(k) = n for each $k \in Z$.

□

(A.1.17) COROLLARY: The columns of Q in (A.1.15) form a basis for the B-module $\mathrm{Im}\Gamma$.

PROOF: At once, we see that $\mathrm{Im}\Gamma = \mathrm{Im}Q$ from relation (A.1.16) and from the fact that Λ is invertible over B. Furthermore, the columns of Q are B-linearly independent.

□

APPENDIX TWO

PARTIAL REALIZATION PROBLEM

In this section, we define a more general realization problem than the one considered in this project. We also find necessary and sufficient conditions for a linear i/o map to be realizable over a proper subset of the causal half plane.

Given a linear input/output map

$$(A.2.1) \qquad f_W : \mathcal{B}_+^m \to \mathcal{B}_+^p : u \to f_W(u) : k \to \sum_{l=-\infty}^{k} W(k, l)u(l)$$

we observe that f_W can be thought of as the input/output description of a linear system P with unit-pulse response matrix $W(k, l)$ which may not be finite-dimensional. Then, the (i,j)th-entry of W, $\omega_{ij}(k, l)$ represent the response of P for $k \geq l$ at the i^{th}-output terminal when in the j^{th}-input terminal a unit pulse is applied at time l. Therefore, it follows that in order to have a complete knowledge of $W(\cdot, \cdot)$ on the causal half-plane CP we must not only apply a unit pulse to each input terminal at every $l \in Z$ but also measure the response of the system at each output terminal at every $k \geq l$. From the practical point of view this is very unrealistic because any input/output experiment must have a finite duration.

From the above observation, we have that the unit-pulse response matrix $W(\cdot, \cdot)$ of the response map f_W will be defined, or known, only on a subset Ω of the causal half-plane CP in general. Hence, a more natural definition of the realization problem for discrete time-varying systems is

(A.2.2) DEFINITION: Let $f_W: \mathcal{B}_+^m \to \mathcal{B}_+^p$ be a linear response map whose unit pulse matrix $W(\cdot, \cdot)$ is known only on a subset Ω of CP. We shall say that f_W is realizable on Ω if there exists a finite-deimensional system $\Sigma = (D, E, H)$ over $\mathcal{B}$ such that

$$W(k, l) = W_\Sigma(k, l) \text{ for each } (k, l) \in \Omega \quad ,$$

where $W_\Sigma(k, l) = H(k)\phi(k, l+1)E(l+1)$ is the unit-pulse response matrix of Σ. We shall also say that Σ is a realization of f_W on Ω. A realization is said to be minimal if there is no realization with smaller dimension.

Given an input/output map f_W as in definition (A.2.2), let us consider the response map $f_{\overline{W}}: \mathcal{B}_+^m \to \mathcal{B}_+^p$ whose unit-pulse response is defined by

$$\text{(A.2.3)} \qquad \overline{W}(k, l) = \begin{cases} W(k, l) & , \quad (k, l) \in \Omega \quad ; \\ 0 & , \quad (k, l) \in CP/\Omega \quad . \end{cases}$$

Then we have at once the following result.

(A.2.4) THEOREM: A linear i/o map $f_W: \mathcal{B}_+^m \to \mathcal{B}_+^p$ is realizable on a subset Ω of CP if and only if the response map $f_{\bar{W}}: \mathcal{B}_+^m \to \mathcal{B}_+^p$ defined by (A.2.3) is realizable on CP.

PROOF: Sufficiency follows directly from definition (A.2.2) and from (A.2.3). Hence necessity only needs being shown. So, assume that f_W is realizable on Ω, then by definition there exists a finite-dimensional system $\Sigma = (D, E, H)$ such that

$$W(k, l) = H(k)\phi_\Sigma(k, l+1)E(l+1) \quad \text{for all } (k, l) \in \Omega \quad .$$

Define the following subsets of Z:

$$\Omega_1 := \{k \in Z: (k, l) \in \Omega, \text{ for some } l \in Z\} \quad ,$$

$$\Omega_2 := \{l \in Z: (k, l) \in \Omega, \text{ for some } k \in Z\} \quad .$$

Let $\bar{\Sigma} = (\bar{D}, \bar{E}, \bar{H})$ be a discrete time-varying system given by

(A.2.5) $\bar{D} = D$.

$$\text{(A.2.6)} \qquad \bar{E}(k) = \begin{cases} E(k), & \text{for each } k \text{ such that } k\text{-}1 \in \Omega_2 \ ; \\ 0, & \text{otherwise} \ . \end{cases}$$

$$\text{(A.2.7)} \qquad \bar{H}(k) = \begin{cases} H(k), & k \in \Omega_1 \ ; \\ 0, & \text{otherwise} \ . \end{cases}$$

From (2.3.6), we know that the unit-pulse response matrix $W_{\bar{\Sigma}}(\cdot, \cdot)$ of $\bar{\Sigma}$ is given by

$$\text{(A.2.8)} \qquad W_{\bar{\Sigma}}(k, l) = \bar{H}(k)\phi_{\bar{\Sigma}}(k, l+1)\bar{E}(l+1) \quad \text{for each} \quad (k, l) \in CP \ .$$

Now, because of (A.2.5) we may write (A.2.8) as

$$\text{(A.2.9)} \qquad W_{\bar{\Sigma}}(k, l) = \bar{H}(k)\phi_{\Sigma}(k, l)\bar{E}(l+1) \ , \quad (k, l) \in CP \ .$$

Suppose $(k, l) \in \Omega$, then we have that $k \in \Omega_1$ and $l \in \Omega_2$. Therefore, from (A.2.6) and (A.2.7) we obtain

$$W_{\bar{\Sigma}}(k, l) = H(k)\phi_{\Sigma}(k, l+1)E(l+1)$$

$$= W(k, l) \quad \text{(by assumption)} \ .$$

On the other hand, if $(k, l) \notin \Omega$, then either $k \notin \Omega_1$ or $l \notin \Omega_2$. Assume that $k \notin \Omega_1$, in that case $\overline{H}(k) = 0$ by definition. When $l \notin \Omega_2$, from (A.2.6) it is seen that $\overline{E}(l+1) = 0$. Hence, we may conclude from (A.2.9) that

$$W_{\overline{\Sigma}}(k, l) = 0 \quad \text{for each } (k, l) \notin \Omega \quad .$$

In resume, we have shown that

$$W_{\overline{\Sigma}}(k, l) = \begin{cases} W(k, l) & , \ (k, l) \in \Omega \\ 0 & , \ \text{otherwise} \end{cases} ,$$

i.e., $\overline{\Sigma}$ is a realization of $f_{\overline{W}}$ on CP.

□

(A.2.10) COROLLARY: Let $\Sigma = (D, E, H)$ be a minimal realization of f_W on Ω, and let $\overline{\Sigma} = (D, E, H)$ be a minimal realization of $f_{\overline{W}}$ on CP. Then

$$\dim \Sigma = \dim \overline{\Sigma} \quad .$$

PROOF: It follows directly from the proof above. □

Let k_0 be a fixed integer and let Ω be defined by

(A.2.11) $\quad \Omega := \{(k, l) \in ZxZ : k > l > k_0\} \quad .$

Note that Ω is a proper subset of the causal half-plane.

Any response map $f_W: B_+^m \to B_+^p$ whose unit-pulse response $W(\cdot, \cdot)$ is known only on Ω (defined in (A.2.11)) may represent the i/o data of a physical system P which underwent an input/output experiment at time $k = k_0$. For this case, the question of realizability is answered by the following.

(A.2.12) THEOREM: Let $f_W: B_+^m \to B_+^p$ be a linear i/o map with unit pulse response $W(\cdot, \cdot)$ defined only on Ω given by (A.2.11). f_W is realizable on Ω if and only if there exists a finite-positive integer n such that for each $k > k_0$

$$(A.2.13) \quad \text{rank} \begin{bmatrix} W(k, k-1) & W(k, k-2) & \cdots & W(k, k_0) \\ W(k+1,k-1) & W(k+1,k-2) & \cdots & W(k+1,k_0) \\ W(k+2,k-1) & W(k+1,k-2) & \cdots & W(k+2,k_0) \\ \vdots & \vdots & & \vdots \end{bmatrix} \leq n,$$

Furthermore, the smallest positive integer n for which (A.2.13) holds is the dimension of a minimal realization.

PROOF: By theorem (A.2.4) f_W is realizable on Ω if and only if $f_{\overline{W}}$, see (A.2.3) is realizable on CP. But from theorem (3.2.15) $f_{\overline{W}}$ is realizable iff there exists a finite positive integer n such that

(A.2.14) $\quad \operatorname{rank} B_{\overline{W}}(k) \leq n \quad \text{for all} \quad k \in Z \quad .$

Recall that

$$B_{\overline{W}}(k) = [\overline{W}_{ij}(k)]_{\lambda<i,j<\infty} \quad , \quad k \in Z$$

where

$$\overline{W}_{ij}(k) := \overline{W}(k+i-1,\ k-j) \quad , \quad i, j \in Z^{+}, \quad k \in Z \quad .$$

Therefore, by using the fact that

$$\overline{W}(k,\ l) = \begin{cases} W(k,\ l) & , \quad k \geq l \geq k_0 \ , \\ \\ 0 & , \quad \text{otherwise} \quad , \end{cases}$$

it is not hard to see that

$$(A.2.15)\quad B_{\overline{W}}(k) = \begin{bmatrix} W(k,k-1) & W(k,k-2) & \cdots & W(k,l_0) & 0 & 0 & \cdots \\ W(k+1,k-1) & W(k+1,k-2) & \cdots & W(k+1,k_0) & 0 & 0 & \cdots \\ W(k+2,k-1) & W(k+2,k-2) & \cdots & W(k+2,k_0) & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix}$$

for each $k \in \{k_0+1,\ k_0+2,\ \ldots\}$.

But, it must be clear that $\text{rank}[A(k) \,;\, 0] = \text{rank } A(k)$ for each $k \in Z$, so from (A.2.14) and (A.2.15) the first part of the theorem follows automatically. The last part to be shown is a direct consequence of theorem (3.2.15) and corollary (A.2.10.).

□

The result of theorem (A.2.12) has been proved to be useful in deciding when the matrix covariance sequence $R_{k,l}$ associated with some (m-vector) random process $\{y_t\}_{t=0}^{\infty}$ is semiseparable (see SCHWARTZ, DICKINSON and SONTAG [1983] for details).

The "dual" of theorem (A.2.12) is the following.

(A.2.16) THEOREM: A linear i/o map $f_W: B_+^m \to B_+^p$ with unit pulse response $W(\cdot,\cdot)$ defined only on the set $\Omega = \{(k, l) \notin Z \times Z : k_0 > k > l,\ k_0 = \text{fixed integer}\}$ is realizable (on Ω) if and only if there exists a finite positive integer n such that for each $k < k_0$

$$\text{(A.2.16)}\quad \text{rank} \begin{bmatrix} W(k,k-1) & W(k,k-2) & W(k,k-3) & \cdots \\ W(k+1,k-1) & W(k+1,k-2) & W(k+1,k-3) & \cdots \\ \vdots & \vdots & \vdots & \\ W(k_0-1,k-1) & W(k_0-1,k-2) & W(k_0-1,k-3) & \cdots \\ W(k_0,k-1) & W(k_0,k-2) & W(k_0,k-3) & \cdots \end{bmatrix} \leq n \ .$$

Furthermore, the smallest positive integer n for which (A.2.16) holds is the dimension of a minimal realization.

PROOF: Use duality in the proof of theorem (A.2.12).

□

Unfortunately, the kind of input/output maps considered in theorems (A.2.12) and (A.2.15) cannot have finite constant rank (even in the case when they are realizable). Therefore, the algorithm developed in this work can not be applied to find canonical realizations of those i/o maps.

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